Finite Element Methods in Solid Mechanics

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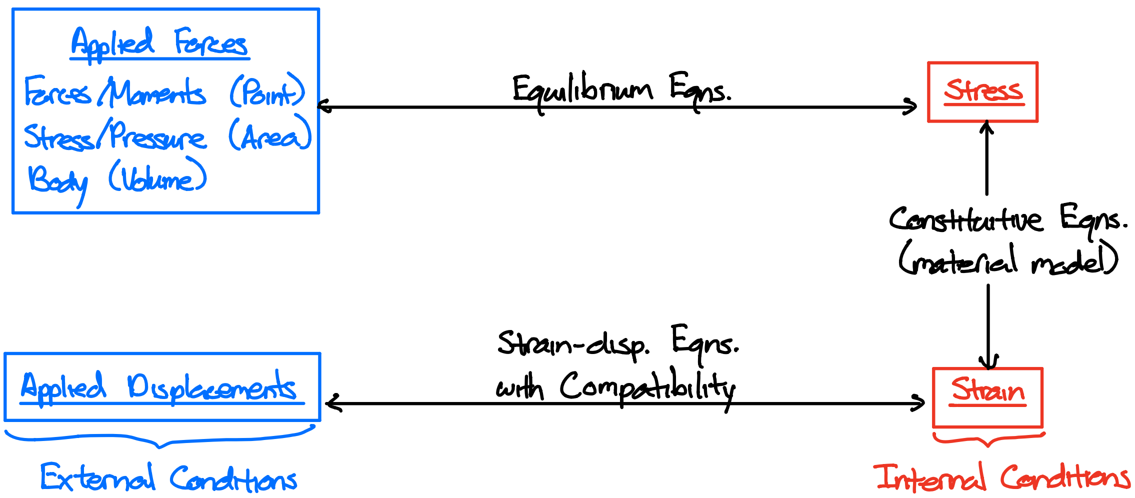
TOPIC 0:
Review of Solid Mechanics

In a typical statics course, we sum the forces and moments over a rigid body in order to determine the equilibrium conditions.

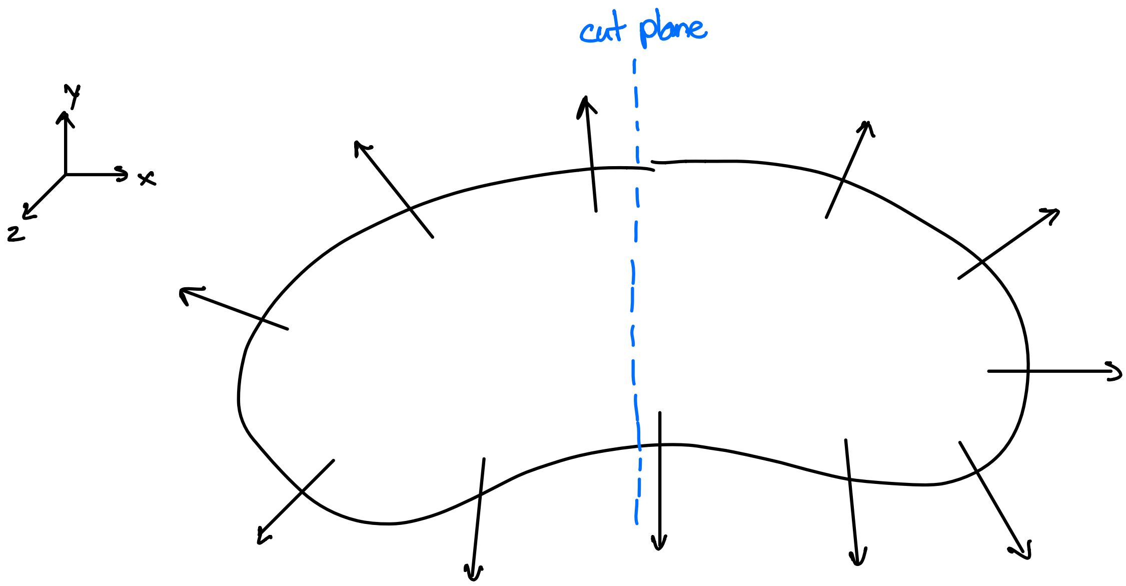
Solid mechanics can be considered an extension of statics where we take into account the internal behavior of a deformable material.

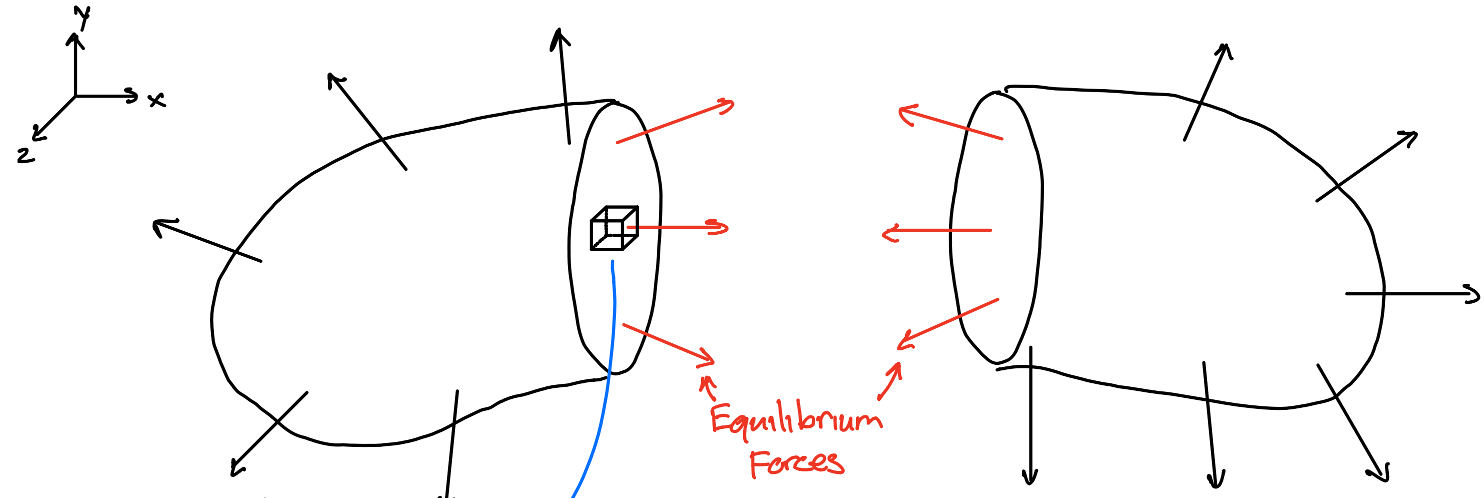
The deformable bodies of solid mechanics are described by PDEs which are easily solved for only very simple cases (e.g., 1D deformation, plane stress [pressure vessels]). For more complicated scenarios, approximate solutions are sought, e.g., the FEM.

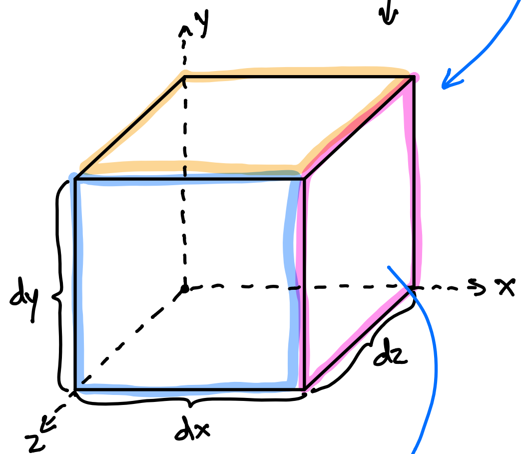


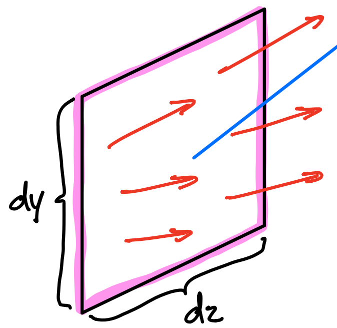
Tonti Diagram

This diagram maps out the solution path for problems in elasticity.









$\bar{F}_x$: force resultant on yz-plane (x-axis normal)

$\bar{F}$ is a function of area A

$$\bar{F} = \int \underbrace{\frac{\partial \bar{F}}{\partial A}}_{\bar{T}} \, dA$$ total force on area element

$\bar{T}$: stress traction

$$(\bar{T}_x)_x = \left(\frac{\partial F_x}{\partial A}\right)_x = \sigma_{xx} = \sigma_x \qquad (\bar{T}_x)_y = \left(\frac{\partial F_x}{\partial A}\right)_y = \sigma_{xy} = \tau_{xy} \qquad (\bar{T}_x)_z = \left(\frac{\partial F_x}{\partial A}\right)_z = \sigma_{xz} = \tau_{xz}$$

$$(\bar{T}_y)_x = \left(\frac{\partial F_y}{\partial A}\right)_x = \sigma_{yx} = \tau_{yx} \qquad (\bar{T}_y)_y = \left(\frac{\partial F_y}{\partial A}\right)_y = \sigma_{yy} = \sigma_y \qquad (\bar{T}_y)_z = \left(\frac{\partial F_y}{\partial A}\right)_z = \sigma_{yz} = \tau_{yz}$$

$$(\bar{T}_z)_x = \left(\frac{\partial F_z}{\partial A}\right)_x = \sigma_{zx} = \tau_{zx} \qquad (\bar{T}_z)_y = \left(\frac{\partial F_z}{\partial A}\right)_y = \sigma_{zy} = \tau_{zy} \qquad (\bar{T}_z)_z = \left(\frac{\partial F_z}{\partial A}\right)_z = \sigma_{zz} = \sigma_z$$

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \rightarrow \begin{bmatrix} \bar{T}_x \\ \bar{T}_y \\ \bar{T}_z \end{bmatrix}$$ (stress state)

$$(\bar{F}_x)_x = P_x = \iint \overbrace{\left(\frac{\partial F_x}{\partial A}\right)_x}^{\sigma_x} dA$$

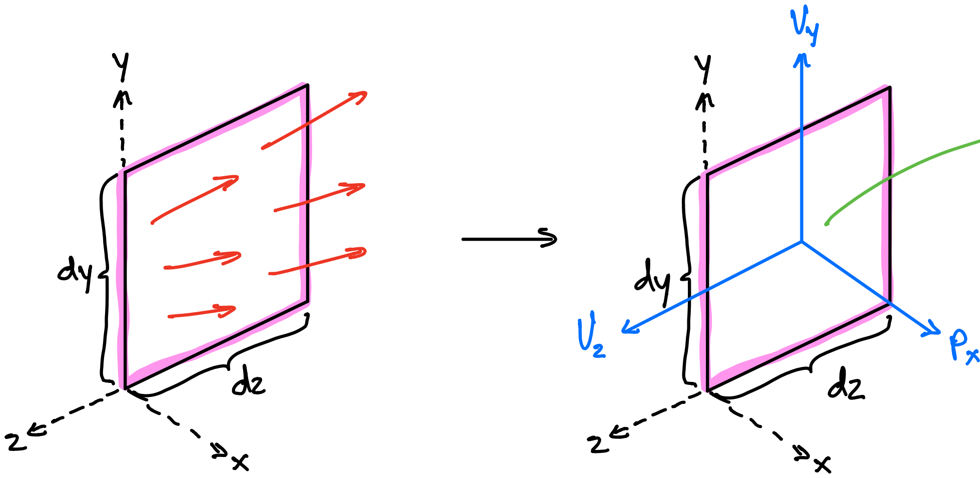
P_x ↳ total force normal to yz-plane

$$(\bar{F}_x)_y = V_y = \iint \overbrace{\left(\frac{\partial F_x}{\partial A}\right)_y}^{\tau_{xy}} dA$$

V_y ↳ total (shear) force parallel to yz-plane and in y-direction

$$(\bar{F}_x)_z = V_z = \iint \overbrace{\left(\frac{\partial F_x}{\partial A}\right)_z}^{\tau_{xz}} dA$$

V_z ↳ total (shear) force parallel to yz-plane and in z-direction



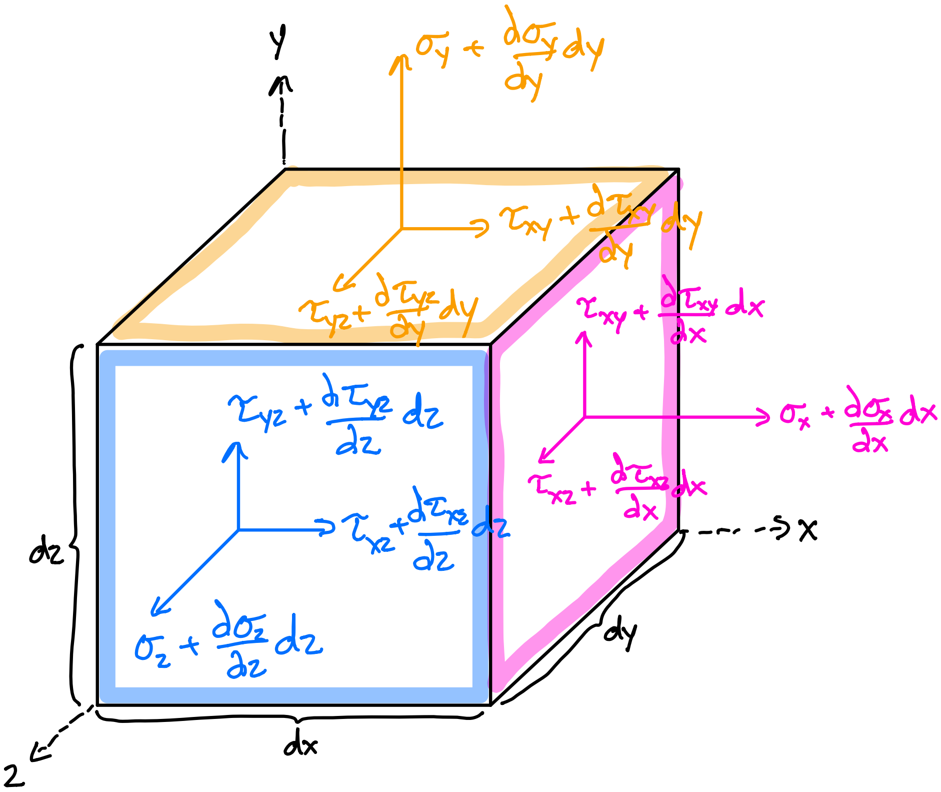
The normal force (P_x) and shear forces (V_y, V_z) act throughout the surface, but we draw them acting through the center just for easy drawing.

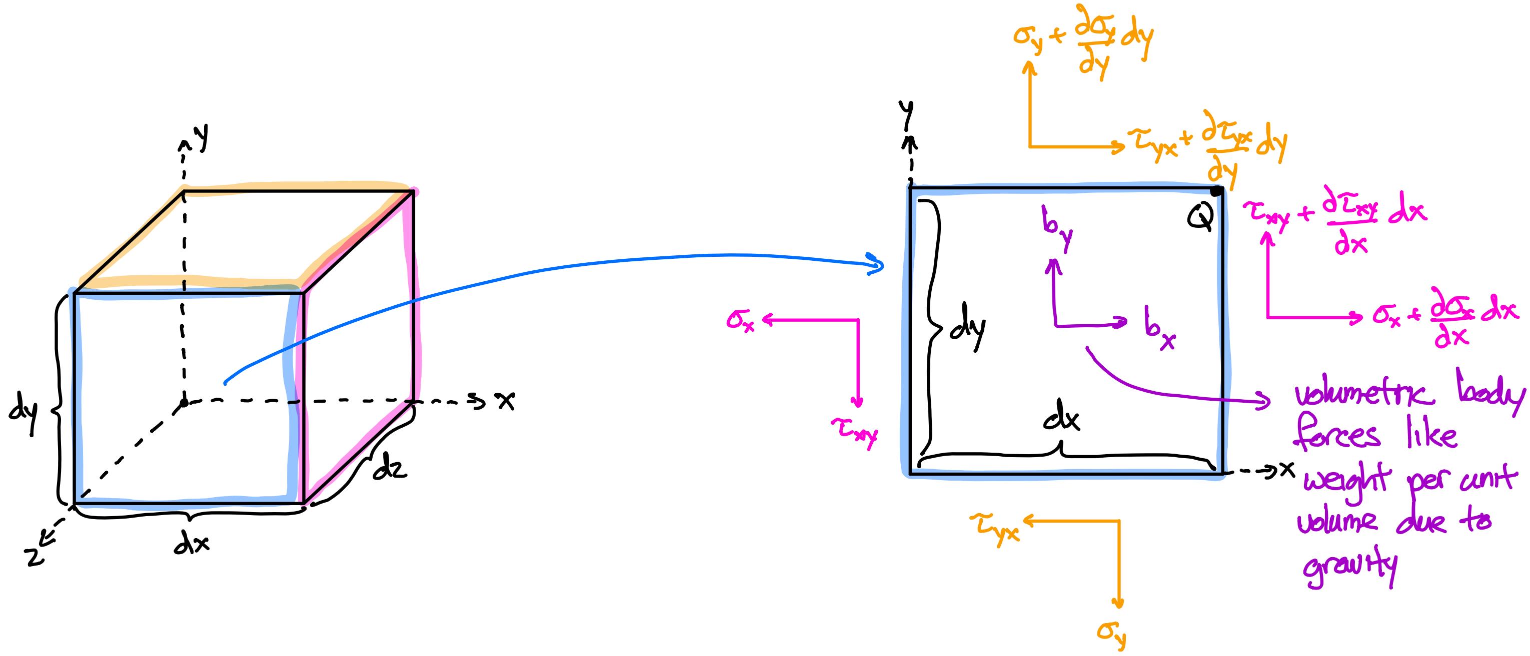
$$M_z = \int (d\bar{F}_x)_x \, y = \int [\sigma_x \, dA] y = \int \sigma_x y \, dA$$

$$M_y = \int (d\bar{F}_x)_x \, z = \int [\sigma_x \, dA] z = \int \sigma_x z \, dA$$

$$M_x = \int (dF_x)_y \, z + \int (dF_x)_z \, y$$

$$= \int [\tau_{xy} dA] z + \int [\tau_{xz} dA] y = \int (\tau_{xy} z + \tau_{xz} y) \, dA$$





$$\Sigma F_x = -\sigma_x\,dydz + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right)dydz - \tau_{yx}\,dxdz + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right)dxdz - \tau_{zx}\,dxdy + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}dz\right)dxdy$$
$$+\, b_x\,dxdydz = 0$$

$$= \frac{\partial \sigma_x}{\partial x}dxdydz + \frac{\partial \tau_{yx}}{\partial y}dxdydz + \frac{\partial \tau_{zx}}{\partial z}dxdydz + b_x\,dxdydz = 0$$

$$\left.\begin{aligned}
&= \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0 \\
\Sigma F_y &= \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + b_y = 0 \\
\Sigma F_z &= \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0
\end{aligned}\right\}\ \text{Equilibrium Eqns.}$$

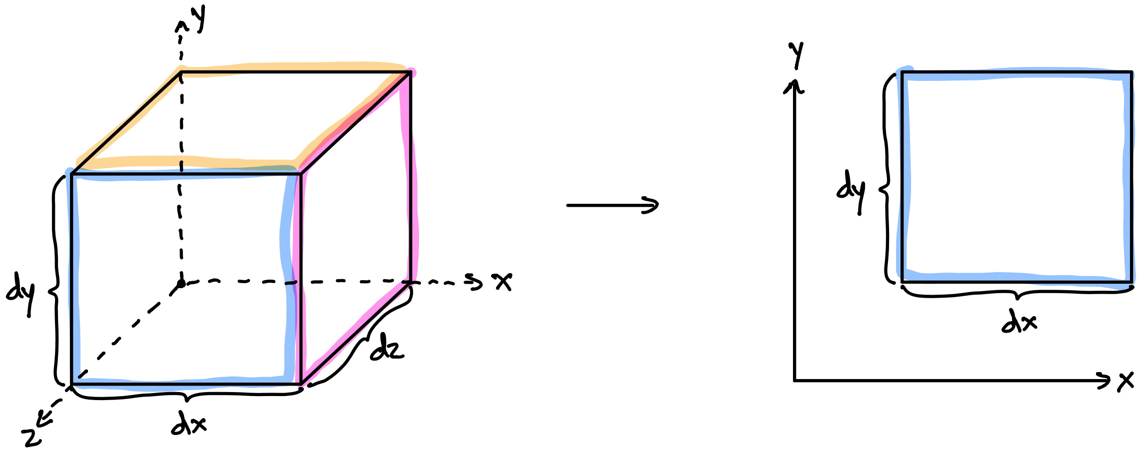
$$\Sigma M_Q = -\sigma_x\,dydz\frac{dy}{2} + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right)dydz\frac{dy}{2} - \tau_{yx}\,dxdz\,dy + \tau_{xy}\,dydz\,dx + \sigma_y\,dxdz\frac{dx}{2} - \left(\sigma_y + \frac{\partial \sigma_y}{\partial y}dy\right)dxdz\frac{dx}{2}$$
$$-\tau_{zx}\,dxdy\frac{dy}{2} + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}dz\right)dxdy\frac{dy}{2} + b_x\,dxdydz\frac{dy}{2} - b_y\,dxdydz\frac{dx}{2} = 0$$

$$= \frac{1}{2}\left(\cancelto{\approx 0}{\frac{\partial \sigma_x}{\partial x}dxdy^2dz} - \cancelto{\approx 0}{\frac{\partial \sigma_y}{\partial y}dx^2dydz} + \cancelto{\approx 0}{\frac{\partial \tau_{zx}}{\partial z}dxdy^2dz} + \cancelto{\approx 0}{b_x dxdy^2dz}\right) - \cancelto{\approx 0}{b_y dx^2dydz} + (\tau_{xy} - \tau_{yx})\,dxdzdy = 0$$

$$\therefore \tau_{xy} = \tau_{yx}$$

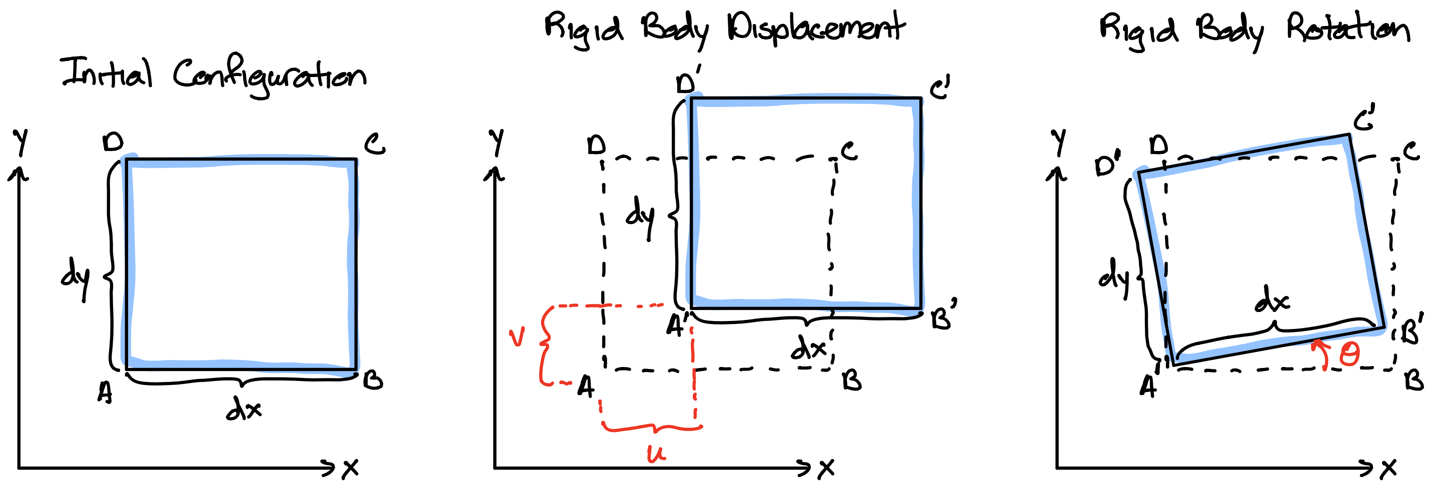
$$\underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}}_{D^T} \underbrace{\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}}_{\bar{\sigma}} + \underbrace{\begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}}_{\bar{b}} = 0 \longrightarrow D^T\bar{\sigma} + \bar{b} = 0$$

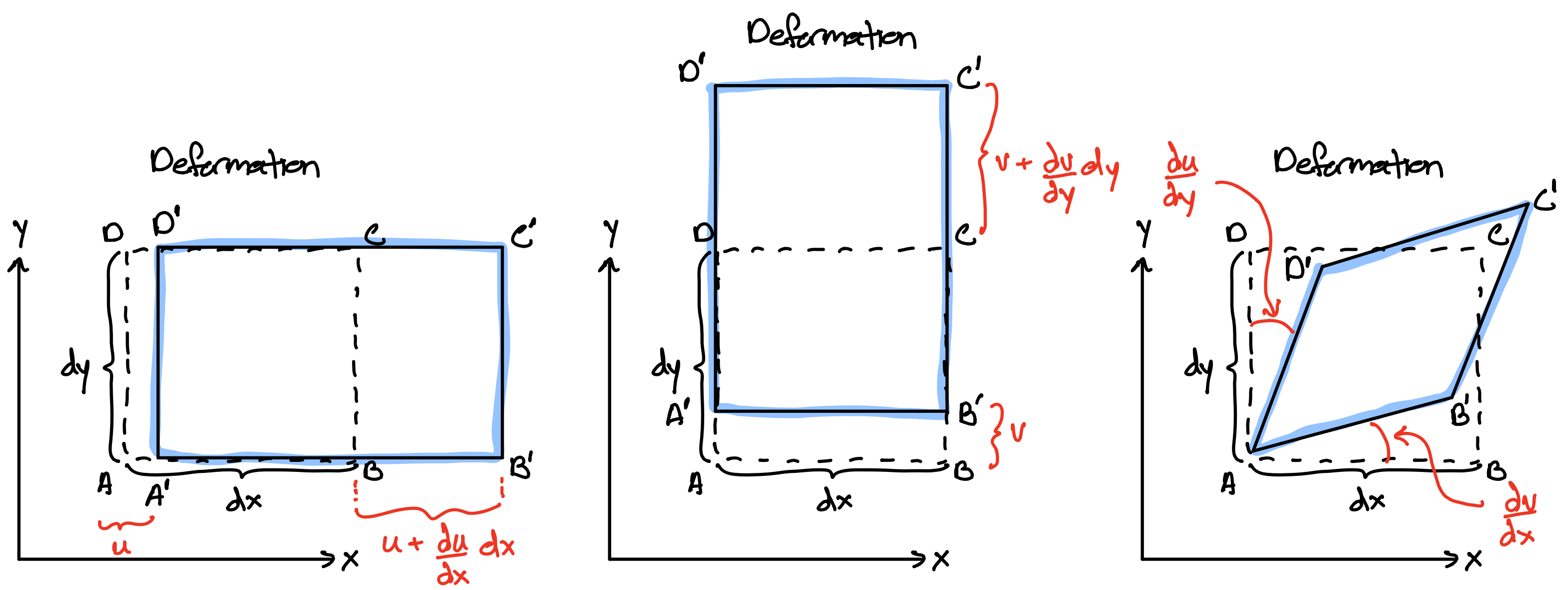
In our study of stress, we never considered the response of the body (e.g., potato, material cube) due to the applied loading. We expect the body to deform, i.e., change size and shape which involves material displacement.



rigid body motion: every bit of material displaces the same amount. The relative displacement is zero.

deformation: bits of material (depending on position) displace a different amount. The relative displacement is non-zero.





normal strain: deformation along (i.e., parallel to) an axis.

$$\varepsilon_x = \frac{u_B - u_A}{dx} = \frac{u + \frac{\partial u}{\partial x}dx - u}{dx} = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{v_D - v_A}{dy} = \frac{v + \frac{\partial v}{\partial y}dy - v}{dy} = \frac{\partial v}{\partial y}$$

shear strain: changes the angle between two originally perpendicular axes.

$$\tan\alpha = \frac{\frac{\partial v}{\partial x}dx}{dx} = \frac{\partial v}{\partial x}$$ by small angle assumption, $\alpha \approx \frac{\partial v}{\partial x}$

$$\tan\beta = \frac{\frac{\partial u}{\partial y}dy}{dy} = \frac{\partial u}{\partial y}$$ by small angle assumption, $\beta \approx \frac{\partial u}{\partial y}$

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Strain is not only elongation/contraction ($\varepsilon_x, \varepsilon_y, \varepsilon_z$) but rotation as well ($\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$).

Stress at a point is determined by 3 perpendicular planes and so is strain.

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad u_1 = u,\ u_2 = v,\ u_3 = w \quad x_1 = x,\ x_2 = y,\ x_3 = z$$

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x} \qquad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\gamma_{xy} = \varepsilon_{21}$$

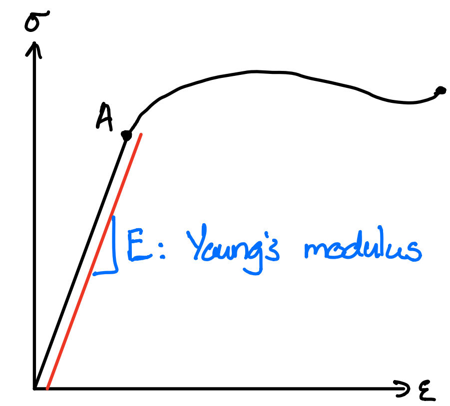
γ_{xy} : engineering shear strain
$\frac{1}{2}\gamma_{xy}$: mathematical shear strain

$$\varepsilon_{ij} = \mathcal{E} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix}$$

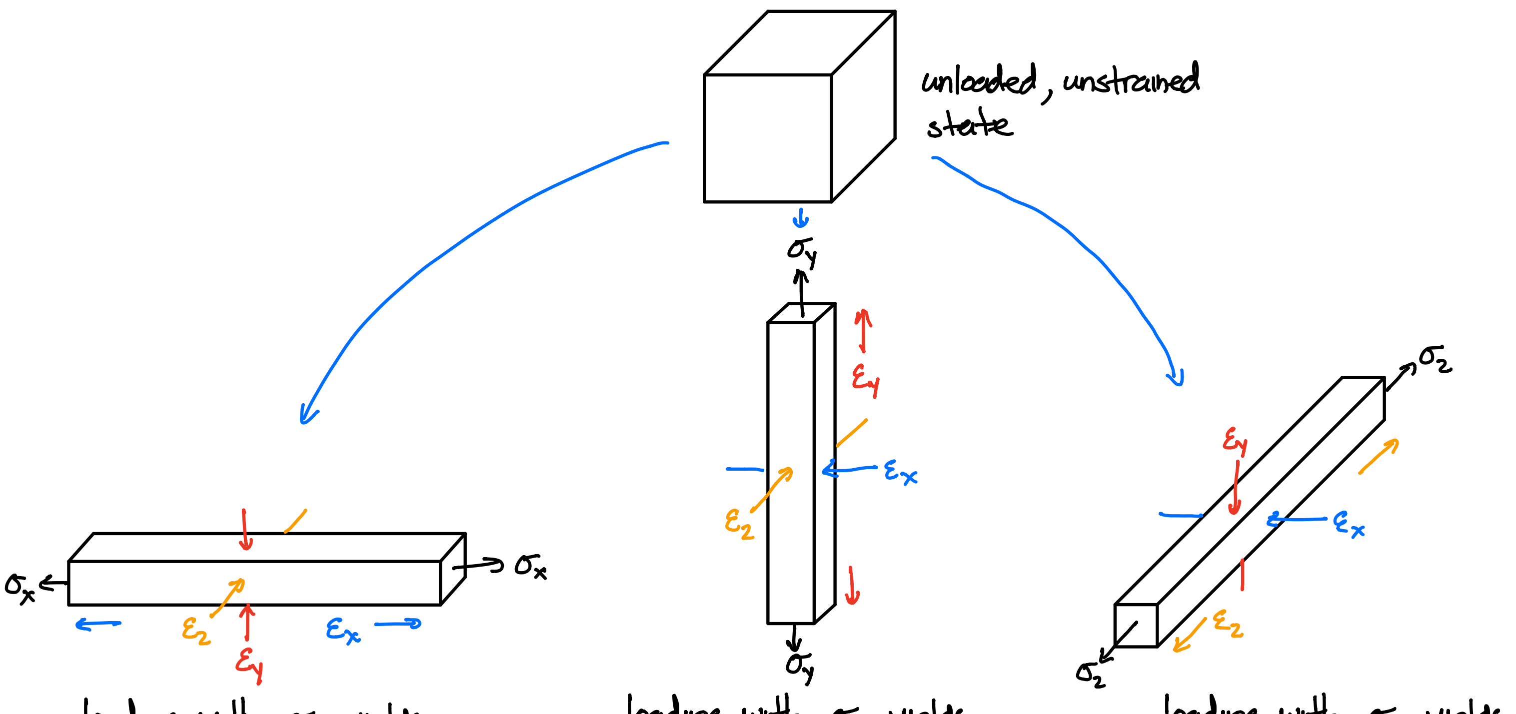
$$\underbrace{\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}}_{\bar{\varepsilon}} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\bar{u}} \longrightarrow \bar{\varepsilon} = D\bar{u}$$

In our study of stress, we deal with the forces on a body. In our study of strain, we deal with the geometric changes of a body. In each case, we made no reference to the material of the body, so those formulas are valid for all materials. But we know from experience that stress and strain are related to each other — these are the constitutive equations.

Most Engineering Materials



(A) Linear limit: stress is a linear function of strain ($\sigma = E\varepsilon$) up to this point. Beyond this point, non-linear effects emerge.



loading with σ_x yields ε_x, but also ε_y and ε_z

loading with σ_y yields ε_y, but also ε_x and ε_z

loading with σ_z yields ε_z, but also ε_x and ε_y

Poissen's ratio, ν, accounts for the effect where an expansion in one direction leads to a contraction in the perpendicular direction.

Loading by σ_x

$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\varepsilon_y = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

$$\varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$$

Loading by σ_y

$$\varepsilon_y = \frac{\sigma_y}{E}$$

$$\varepsilon_x = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E}$$

$$\varepsilon_z = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E}$$

Loading by σ

$$\varepsilon_z = \frac{\sigma_z}{E}$$

$$\varepsilon_x = -\nu\varepsilon_z = -\nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\varepsilon_z = -\nu\frac{\sigma_z}{E}$$

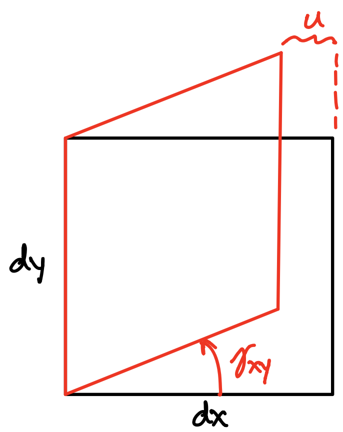
$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \qquad \varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \qquad \varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

In terms of $\sigma(\varepsilon)$:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right]$$



$$du = \varepsilon_x dx = \underbrace{(\cos\gamma_{xy} - 1)}_{\cos\gamma_{xy} \approx 1 \text{ since } \gamma_{xy} \text{ is small}} dx \approx 0$$

$\therefore \varepsilon_x \approx 0$, normal strains do not affect shear strains

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

3D Isotropic Constitutive Relations:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \underbrace{\frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}}_{\text{Elasticity (Constitutive) Matrix}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

Young's modulus (E): measures material resistance to normal loads.

Shear modulus (G): measures material resistance to shear loads.

$$G = \frac{E}{2(1+\nu)} \qquad 0 < E < \infty \qquad -1 < \nu < 1/2$$

Problems involving the full 3D constitutive equations are often difficult to solve without the use of numerical tools. Conditions of plane stress and plane strain allow the analysis to be simplified and applied to 2D problems.

plane stress: one dimension much smaller than the other two: thin plate, thin walled pressure vessel, thin disk, gears. All loading is in the plane.

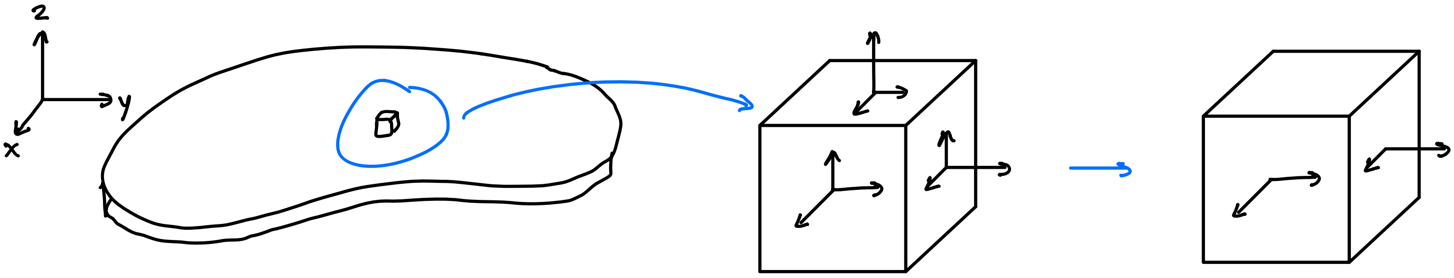
$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \tau_{xy} = G\gamma_{xy}$$

$$\longrightarrow \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}\frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0\\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0\\ 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$$

$$\varepsilon_{zz} = -\frac{\nu}{E}(\sigma_x + \sigma_y) \qquad \gamma_{xz} = \gamma_{yz} = 0 \qquad \sigma_z = \tau_{xz} = \tau_{yz} = 0$$



plane strain: one dimension much larger than the ot er two: tubes shafts under pressure, retaining walls or dams.

$$\varepsilon_x = \frac{1-\nu^2}{E}\left(\sigma_x - \frac{\nu}{1-\nu}\sigma_y\right) \qquad \varepsilon_y = \frac{1-\nu^2}{E}\left(\sigma_y - \frac{\nu}{1-\nu}\sigma_x\right) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

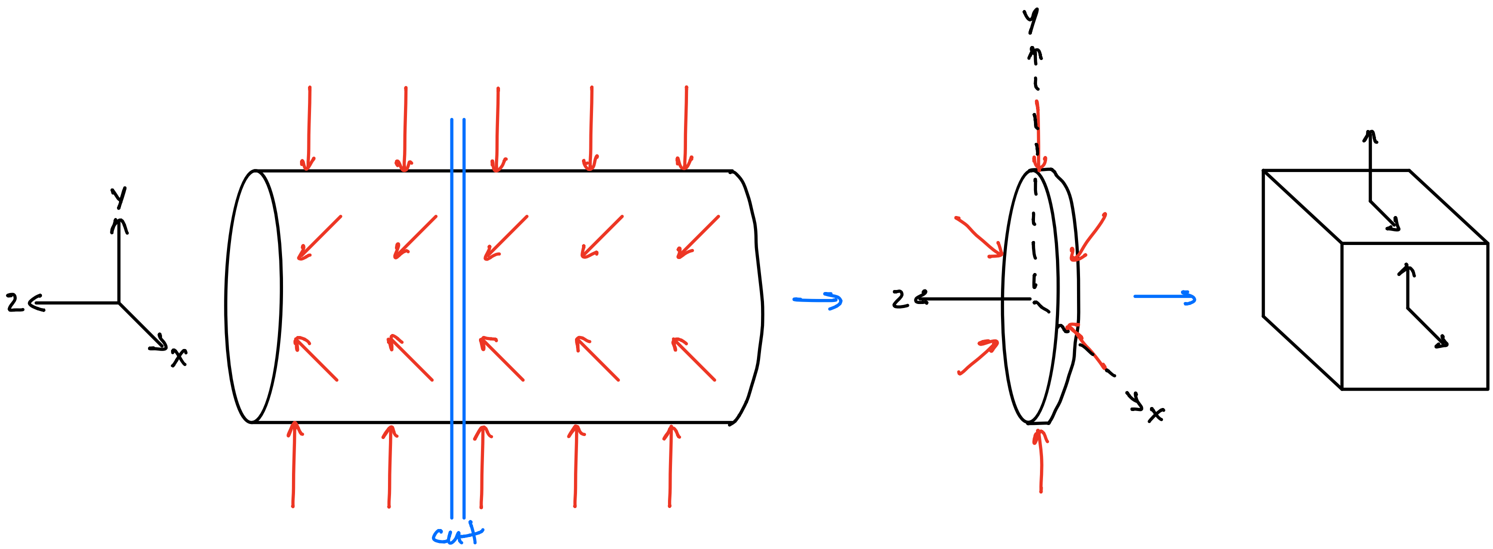
$$\sigma_x = \frac{\nu E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu\varepsilon_y]$$

$$\sigma_y = \frac{\nu E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu\varepsilon_x]$$

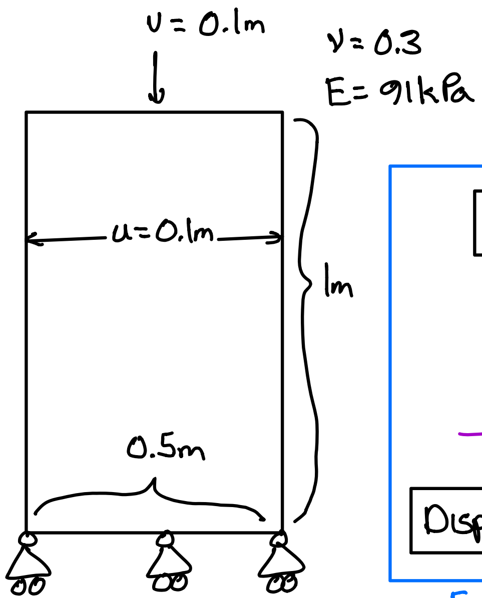
$$\tau_{xy} = G\gamma_{xy}$$

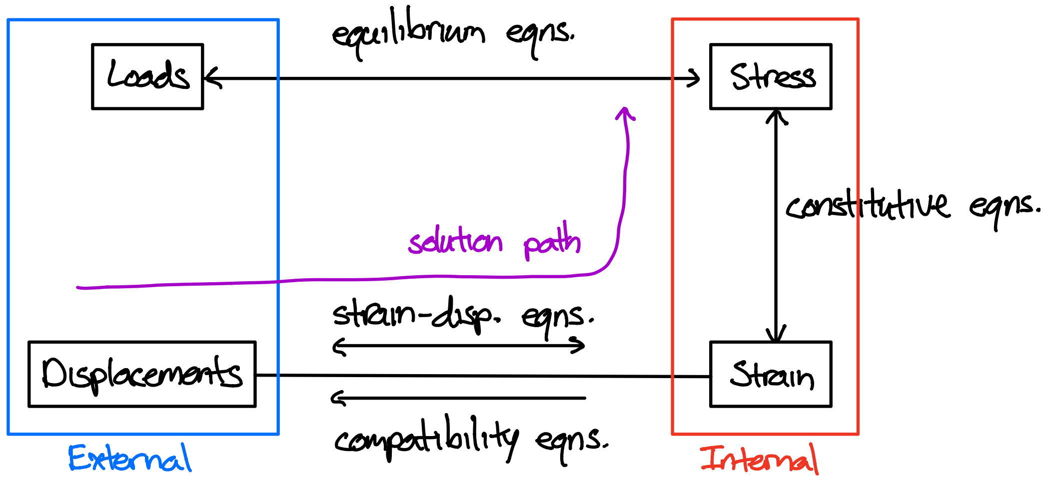
$$\sigma_z = \frac{\nu E}{(1+\nu)(1-2\nu)}(\varepsilon_x + \varepsilon_y) \qquad \tau_{xz} = \tau_{yz} = 0$$

$$\longrightarrow \begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & G\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{bmatrix}$$



Given the thin plate with the dimensions and material properties shown below, in response to applied displacements, what are the internal stresses?





$$\varepsilon_x = \frac{du}{dx} = \frac{0.1m}{0.5m} = 0.2$$

$$\varepsilon_y = \frac{dv}{dy} = -\frac{0.1m}{1.0m} = -0.1$$

plane stress

$$\therefore \sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = 17kPa$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = -4kPa$$

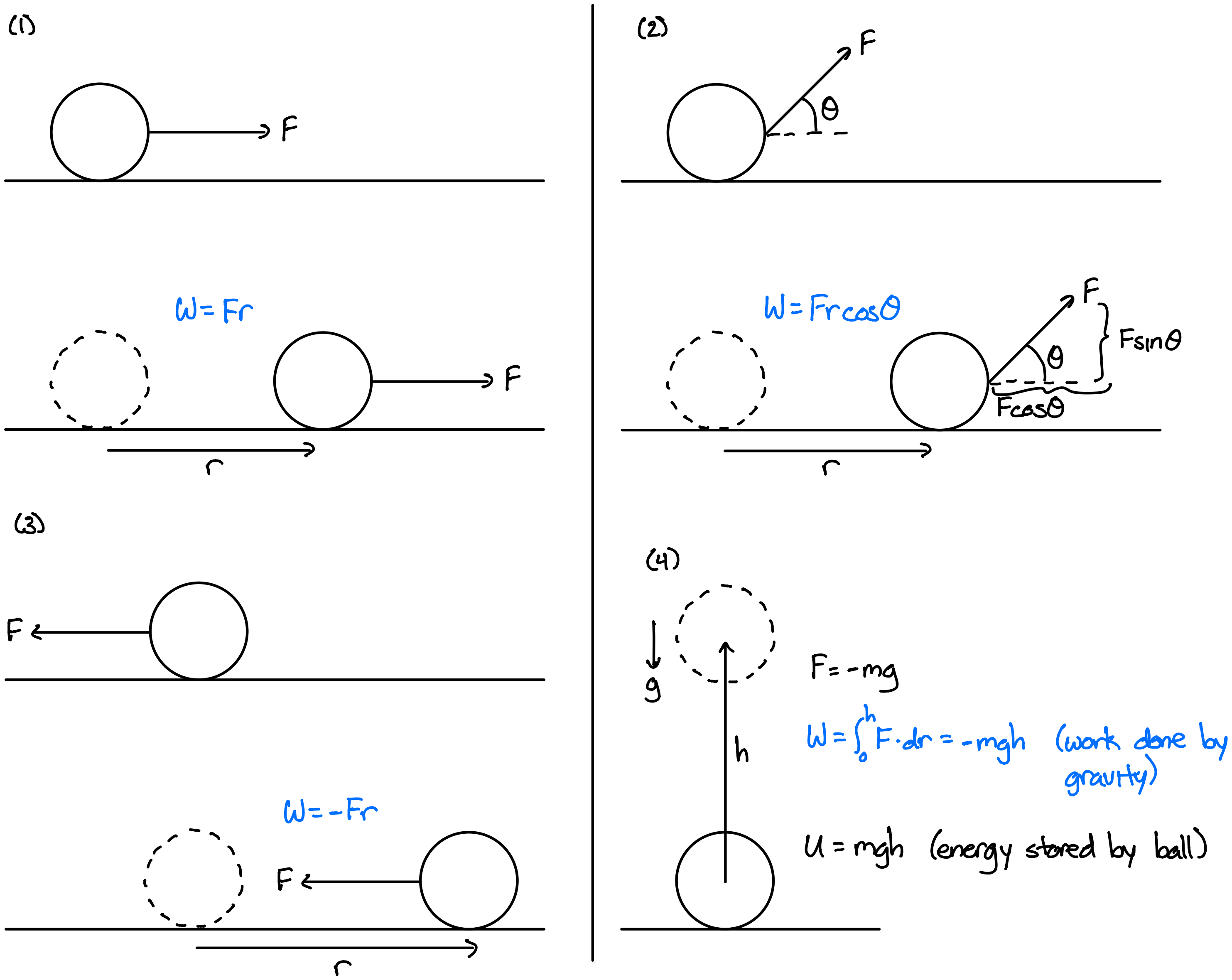
TOPIC 1:
Energy, Virtual Work, and Minimum Potential Energy

energy: the fundamental mechanism by which a system may be altered in some way.

kinetic energy: energy of motion

potential energy: energy position/configuration (gravity, electric/magnetic fields, strain)

work: energy supplied to/removed from a system by external factors; the product of a force and a simultaneous, in-line displacement: $W = \int \bar{F} \cdot d\bar{r}$



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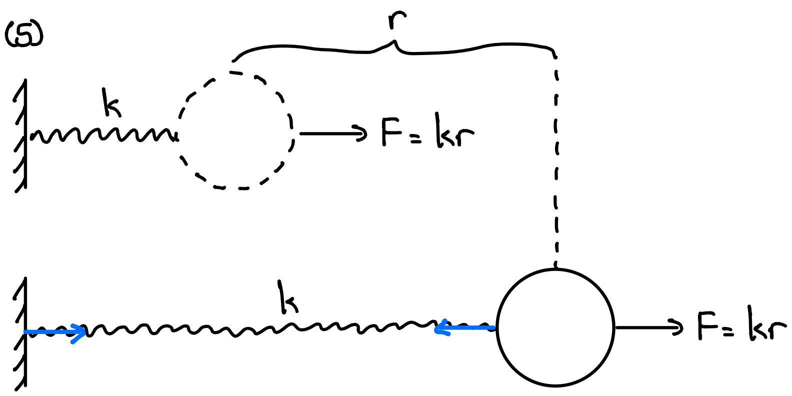


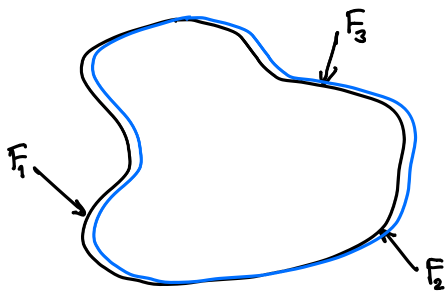
$F = \underline{kr}$ (force you apply to mass)

$\underline{W} = \int \bar{F} \cdot d\bar{r} = \int kr\,dr = \frac{1}{2}kr^2 = \frac{1}{2}Fr = \underline{U}$

$F = -\frac{dU}{dr} = -kr$ (force spring applies to mass)

The principle of virtual work allows the governing equations of a system to be derived from an energy perspective and aids in discovering approximate solutions. We will use this as a part of the minimum potential energy theorem as a basis for developing the FEM. It is, essentially, a statement of static equilibrium.

We cannot talk about virtual work without defining virtual displacements. Consider an rigid body acted upon by an arbitrary set of forces/moments. Imagine that the body displaces without changing any of the forces/moments that act upon it.



We're free to imagine any displacement we want as long as the virtual displacements obey a set of rules:

1. infintesimal (behave as differentials) $du \equiv \delta u$ imagined differential, actual differential

2. instantaneous (imagined displacements do not imply the passage of time)

3. kinematically admissable (must not violate constraint equations)

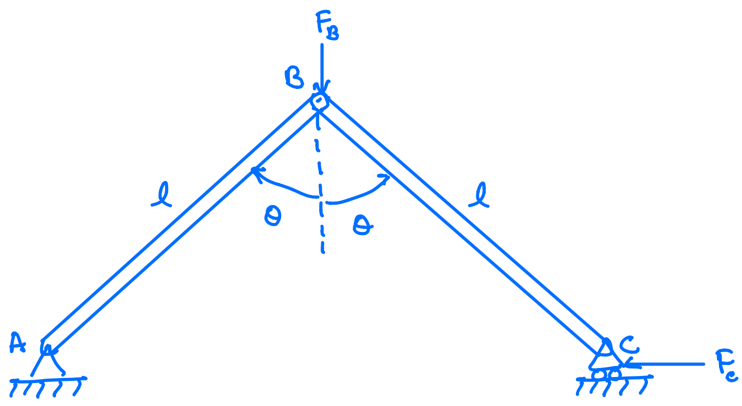
Since the forces are present during the virtual displacements, some virtual work is done:

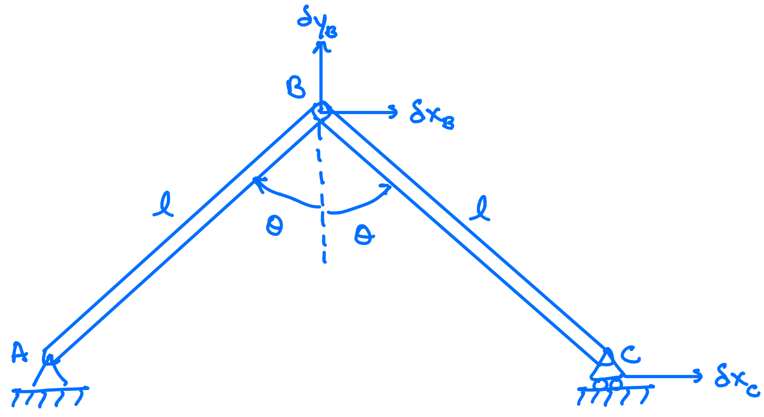
$$\delta W = \bar{F}_1 \cdot \delta\bar{u}_1 + \bar{F}_2 \cdot \delta\bar{u}_2 + \cdots + \bar{F}_n \cdot \delta\bar{u}_n = \left(\sum_{i=1}^{n} \bar{F}_i\right) \cdot \delta\bar{u}_i$$

Since δu_i is as arbitrary as our imagination (with the above constraints), for a rigid body to be in static equilibrium, $\delta W = 0$ since $\sum_{i=1}^{n} \bar{F}_i = 0$.

For rigid bodies, the principle of virtual work offers no advantages over Newton's 2nd law from statics. However, for a system of interconnected bodies, the principle of virtual work simplifies things by allowing us to ignore internal loads and reactions at fixed supports since these do no work.

Problem: Determine the force F_B required to balance the load F_C on an assembly of two pinned-connected rods.





$$\delta W = (0)\delta x_B - F_B\,\delta y_B - F_C\,\delta x_C$$

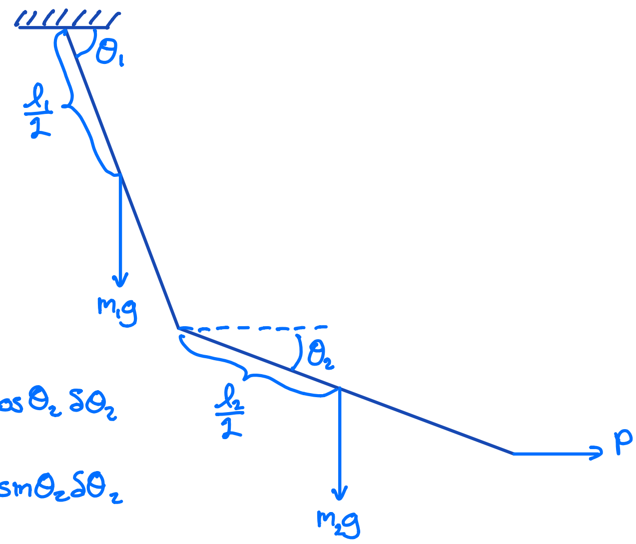
$y_B = l\cos\theta \qquad \delta y_B = -l\sin\theta\,\delta\theta$

$x_C = 2l\sin\theta \qquad \delta x_C = 2l\cos\theta\,\delta\theta$

$\delta W = (-F_B\sin\theta + 2F_C\cos\theta)\,l\,\delta\theta = 0 \qquad \therefore\ 2F_C\cos\theta - F_B\sin\theta = 0 \longrightarrow F_C = \frac{1}{2}F_B\tan\theta$

In statics, we would have to draw free-body diagrams for each member and involve reaction/internal forces. The principle of virtual work greatly simplified things.

Problem: Determine the equilibrium configuration of the double pendulum with horizontal load P applied to the free end.



$$\delta W = m_1 g\,\delta y_1 + m_2 g\,\delta y_2 + P\,\delta x_E = 0$$

$y_1 = \frac{l_1}{2}\sin\theta_1 \qquad \delta y_1 = \frac{l_1}{2}\cos\theta_1\,\delta\theta_1$

$y_2 = l_1\sin\theta_1 + \frac{l_2}{2}\sin\theta_2 \qquad \delta y_2 = l_1\cos\theta_1\,\delta\theta_1 + \frac{l_2}{2}\cos\theta_2\,\delta\theta_2$

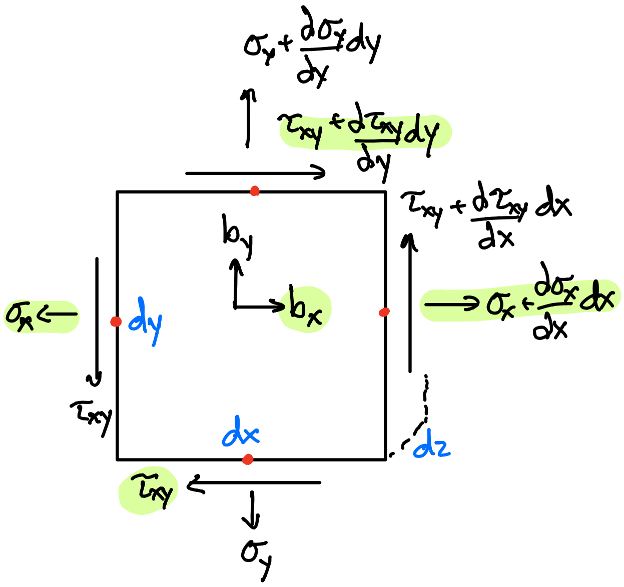
$x_E = l_1\cos\theta_1 + l_2\cos\theta_2 \qquad \delta x_E = -l_1\sin\theta_1\,\delta\theta_1 - l_2\sin\theta_2\,\delta\theta_2$

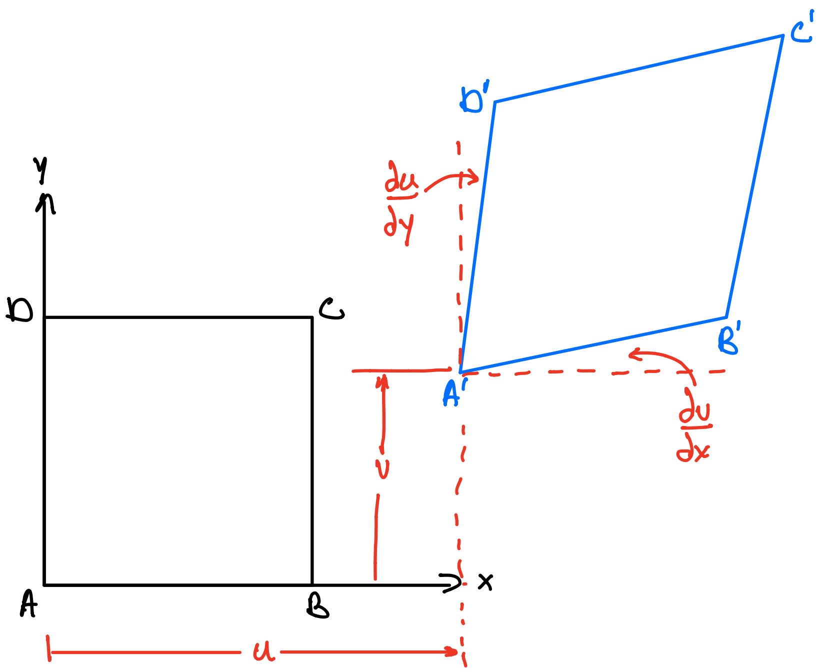
$$\delta W = \frac{l_1}{2}\underbrace{[(m_1+2m_2)g\cos\theta_1 - 2P\sin\theta_1]}_{\text{must vanish}}\delta\theta_1 + \frac{l_2}{2}\underbrace{(m_2 g\cos\theta_2 - 2P\sin\theta_2)}_{\text{must vanish}}\delta\theta_2 = 0$$

$(m_1+2m_2)g\cos\theta_1 - 2P\sin\theta_1 = 0 \qquad \therefore\ \theta_1 = \tan^{-1}\left[\frac{(m_1+2m_2)g}{2P}\right]$

$m_2 g\cos\theta_2 - 2P\sin\theta_2 \qquad \therefore\ \theta_2 = \tan^{-1}\frac{m_2 g}{2P}$

Work Done on Deformable Body





$$dW = -(\sigma_x dy dz)\, d\left(u + \frac{\partial u}{\partial y}\frac{dy}{2}\right) + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) d\left(u + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y}\frac{dy}{2}\right) dy dz$$

$$- (\tau_{xy} dx dz)\, d\left(u + \frac{\partial u}{\partial x}\frac{dx}{2}\right) + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right)(dx dz)\, d\left(u + \frac{\partial u}{\partial x}\frac{dx}{2} + \frac{\partial u}{\partial y} dy\right)$$

$$+ b_x (dx dy dz)\, d\left(u + \frac{\partial u}{\partial x}\frac{dx}{2} + \frac{\partial u}{\partial y}\frac{dy}{2}\right) \qquad \text{x-direction}$$

Note: $d\left(u + \frac{\partial u}{\partial x} dx\right) = du + d\varepsilon_x dx + \varepsilon_x \cancelto{0}{(d\,dx)}$ since dx is constant

$$dW = \left[\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x\right) du + \left(\frac{\partial \sigma_x}{\partial x} + \frac{1}{2}\frac{\partial \tau_{xy}}{\partial y} + \frac{1}{2} b_x\right) d\varepsilon_x dx + \left(\frac{1}{2}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{1}{2} b_x\right) d\left(\frac{\partial u}{\partial y}\right) dy\right.$$

$$\left. + \sigma_x d\varepsilon_x + \tau_{xy} d\left(\frac{\partial u}{\partial y}\right)\right] dx dy dz \qquad \text{x-direction}$$

This is the equilibrium equation in the x-direction. For equilibrium to be maintained, it must vanish.

Differentials represent very small changes in a quantity. Therefore, the product of multiple differentials quickly approaches zero. Above, $d\varepsilon_x dx$ and $d\left(\frac{\partial u}{\partial y}\right) dy$ are much much smaller than $d\varepsilon_x$ and $d\left(\frac{\partial u}{\partial y}\right)$, so much so that they can be ignored without significantly impacting the final result.

$$dW = \left[\sigma_x d\varepsilon_x + \tau_{xy} d\left(\frac{\partial u}{\partial y}\right)\right] \underbrace{dx dy dz}_{dV:\ \text{differential volume}} \qquad \text{x-direction} \qquad \left[\text{if } \frac{\partial u}{\partial y} = 0, \text{ then we have the 1D result}\right]$$

Include work from y-direction forces/displacements:

$$dW = \left[\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \tau_{xy} d\left(\frac{\partial u}{\partial y}\right) + \tau_{xy} d\left(\frac{\partial v}{\partial x}\right)\right] dV$$

$$= \left[\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \tau_{xy} d\gamma_{xy}\right] dV \qquad (2D)$$

Include work from z-direction forces/moments:

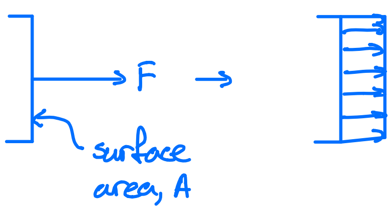
$$dW = \left[\sigma_x d\varepsilon_x + \sigma_y d\varepsilon_y + \sigma_z d\varepsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{xz} d\gamma_{xz} + \tau_{yz} d\gamma_{yz}\right] dV \qquad (3D)$$

Integrating strains gives the total work done. In the following, we consider $d\tilde{\omega} = dW/dV$, the (differential) work per unit volume.

$$\tilde{\omega} = \frac{1}{2}\left(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right) \qquad (3D)$$

Problem: What is the work done by a horizontal load applied to the tip of a cantilever beam?

$$dW = \frac{1}{2}\left(\sigma_x \varepsilon_x + \cancelto{0}{\sigma_y \varepsilon_y} + \cancelto{0}{\sigma_z \varepsilon_z} + \cancelto{0}{\tau_{xy}\gamma_{xy}} + \cancelto{0}{\tau_{xz}\gamma_{xz}} + \cancelto{0}{\tau_{yz}\gamma_{yz}}\right) dV = \frac{1}{2}\sigma_x \varepsilon_x \, dV$$ since this is a 1D problem



$\sigma_x = \dfrac{F}{A}$ average stress (constant)

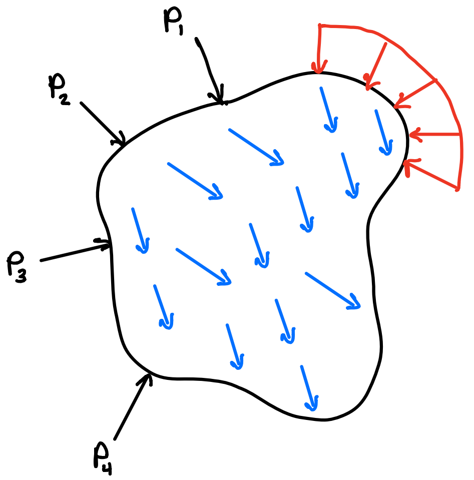
$$W = \frac{1}{2}\int_0^l \int_A \underbrace{\sigma_x \varepsilon_x}_{\varepsilon_x = \frac{\sigma_x}{E}} \, dA\,dx = \frac{1}{2}\int_0^l \int_A \underbrace{\frac{\sigma_x^2}{E}}_{\sigma_x = \frac{F}{A}} \, dA\,dx = \frac{1}{2}\int_0^l \underbrace{\int_A}_{\text{integrate over area from 0 to } A_{tot}} \frac{F^2}{A^2 E} \, dA\,dx = \frac{F^2 l}{2AE}$$

work done by constant force in 1D

Because of the linear force-displacement relationship, $W = \dfrac{F^2 l}{2AE} = \dfrac{1}{2} F \underbrace{\left(\dfrac{Fl}{AE}\right)}_{u\text{: displacement at } l}$

We have considered virtual work from the perspective of rigid bodies, which must vanish in the case that the system is in equilibrium as the applied forces sum to zero. For deformable bodies, however, some energy is stored in deforming the material, which prevents the virtual work from vanishing.

$$U = \frac{1}{2}\int_V (\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz})\, dV = W$$ strain energy of a deformable body

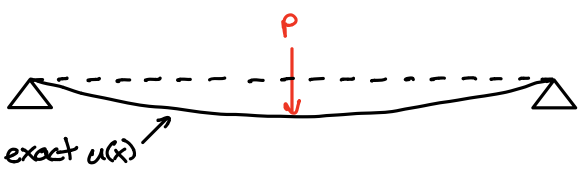


point forces, $\bar{P}$
area (surface) forces, $\bar{T}$
body (volumetric) forces, $\bar{b}$

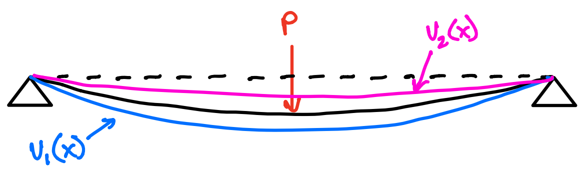
$$U = W = \left(\sum_{i=1}^{n} \bar{P}_i\right)\cdot \bar{u} + \int_A (\bar{T}\, dA)\cdot \bar{u} + \int_V (\bar{b}\, dV)\cdot \bar{u}$$

$$\delta U = \delta W = \left(\sum_{i=1}^{n} \bar{P}_i\right)\cdot \delta\bar{u} + \int_A (\bar{T}\, dA)\cdot \delta\bar{u} + \int_V (\bar{b}\, dV)\cdot \delta\bar{u}$$

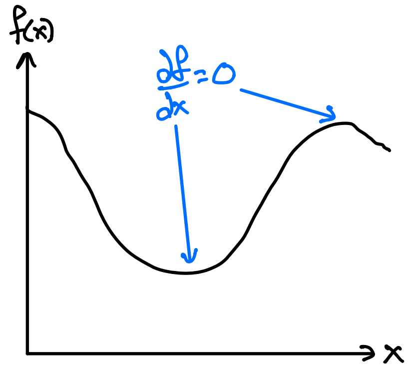
$$\delta \underbrace{\left[U - \left(\sum_{i=1}^{n} \bar{P}_i\right)\cdot u - \int_A (\bar{T}\, dA)\cdot u - \int_V (\bar{b}\, dV)\cdot u\right]}_{\Pi \text{ (total potential energy of a deformable body)}} = 0$$



When a load is applied to an elastic body, the work done by the load balances the potential (strain) energy stored in the body: $U - W = 0$



If we do not know $u(x)$, then we can guess a displacement, $v(x)$, which is a function we make up. If we calculate U and W based on $v(x)$, then we'll find that $U - W \neq 0$. However, we can try to find a <u>form of $v(x)$</u> such that $\Pi = U - W$ is as close to zero as possible... we want to find the form of $v(x)$ that minimizes Π.



For a function, we find the function minima/maxima by locating the point where the gradient vanishes. A similar concept works for finding the minima/maxima of a functional (a function of a function like $\Pi[v]$ which is a function of the displacement function, $v(x)$).

Rayleigh - Ritz Rules:

1. the displacement function may take any continuous form: polynomial, trigonometric, logarithmic, a combination, etc.

2. the displacement function must adhere to kinematic constraints (i.e., known displacements from problem definition must not be violated).

3. more terms in displacement function come with more tuning coefficients; more tuning coefficients allows for greater fine tuning of the final solution which improves accuracy.

4. do not eliminate tuning coefficients unless kinematic constraints require it; otherwise you may worsen the accuracy of your final solution.

Problem: Determine the displacement profile, $u(x)$, for a long, thin cantilever beam with a horizontal tip load. Use the Rayleigh-Ritz Method.

$u = a_0 + a_1 x + a_2 x^2$

$u(0) = a_0 = 0 \qquad \therefore \qquad u(x) = a_1 x + a_2 x^2$

$$U = \frac{1}{2}\int_0^l \left(\cancel{\frac{M^2}{EI}} + \frac{F^2}{AE} + \cancel{\frac{kV^2}{GA}} + \cancel{\frac{T^2}{GJ}} \right) dx = \frac{1}{2}\int_0^l \frac{F^2}{AE}\, dx$$

no bending; no shear; no torsion

$$= \frac{AE}{2}\int_0^l \left(\frac{du}{dx}\right)^2 dx = \frac{AE}{2}\int_0^l (a_1 + 2a_2 x)^2 dx = \frac{AEl}{2}\left[a_1^2 + 2a_1 a_2 l + \frac{4l^2}{3} a_2^2\right]$$

$$\sigma = \frac{F}{A} = E\varepsilon = E\frac{du}{dx}$$

$$F = AE\frac{du}{dx}$$

$W = Pu(l) = Pl(a_1 + a_2 l)$

$\Pi(a_1, a_2) = U - W$

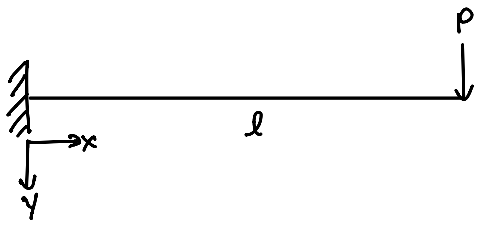
$$\delta\Pi = \frac{\partial \Pi}{\partial a_1}\delta a_1 + \frac{\partial \Pi}{\partial a_2}\delta a_2 = 0$$

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial a_1} &= EAl a_1 + EAl^2 a_2 - Pl = 0 \\ \frac{\partial \Pi}{\partial a_2} &= EAl^2 a_1 + \frac{4}{3}EAl^3 a_2 - Pl^2 = 0 \end{aligned}\right\} \quad a_1 = \frac{P}{EA}; \quad a_2 = 0$$

← extra parameters will go to zero

$$u(x) = \frac{P}{EA}x$$

Problem: Determine the displacement profile, $u(x)$, for a long, thin cantilever beam with a vertical tip load. Use the Rayleigh-Ritz Method.



$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\left.\begin{aligned} u(0) &= a_0 = 0 \\ \left.\frac{du}{dx}\right|_{x=0} &= a_1 = 0 \end{aligned}\right\} \; u(x) = a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\frac{d^2u}{dx^2} = 2a_2 + 6a_3 x + 12 a_4 x^2$$

$$U = \frac{1}{2}\int_0^l \left(\frac{M^2}{EI} + \cancel{\frac{F^2}{AE}} + \cancel{\frac{kV^2}{GA}} + \cancel{\frac{T^2}{GJ}}\right) dx = \frac{1}{2}\int_0^l \frac{M^2}{EI}\, dx$$

no axial loads; no shear; no torsion

$$M = EI \frac{d^2u}{dx^2}$$

$$U = \frac{EI}{2}\int_0^l (2a_2 + 6a_3 x + 12 a_4 x^2)^2 dx$$

$$= \frac{2EIl}{5}\left[5a_2^2 + 5a_2 l(3a_3 + 4la_4) + 3l^2(5a_3^2 + 15 l a_3 a_4 + 12 l^2 a_4^2)\right]$$

$$W = Pu\big|_{x=l} = Pl^2(a_2 + a_3 l + a_4 l^2)$$

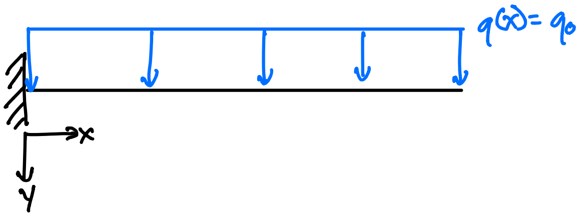
$$\Pi(a_2, a_3, a_4) = U - W \qquad \delta\Pi = \underbrace{\frac{\partial \Pi}{\partial a_2}}_{=0}\delta a_2 + \underbrace{\frac{\partial \Pi}{\partial a_3}}_{=0}\delta a_3 + \underbrace{\frac{\partial \Pi}{\partial a_4}}_{=0}\delta a_4 = 0$$

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial a_2} &= EI(4la_2 + 6l^2 a_3 + 8l^3 a_4) - Pl^2 = 0 \\ \frac{\partial \Pi}{\partial a_3} &= EI(6l^2 a_2 + 12 l^3 a_3 + 18 l^4 a_4) - Pl^3 = 0 \\ \frac{\partial \Pi}{\partial a_4} &= EI\left(8l^3 a_2 + 18 l^4 a_3 + \frac{144 l^5 a_4}{5}\right) - Pl^4 = 0 \end{aligned}\right\} \; a_2 = \frac{Pl}{2EI};\; a_3 = -\frac{P}{6EI};\; a_4 = 0 \leftarrow$$

extra parameters will go to zero

$$\therefore \; u(x) = \frac{Px^2}{6EI}(3l - x)$$

Problem: Determine the displacement profile, $u(x)$, for a long, thin cantilever beam with a vertical distributed load. Use the Rayleigh-Ritz Method.



$$u = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\left.\begin{aligned} u(0) &= a_0 = 0 \\ \left.\frac{du}{dx}\right|_{x=0} &= a_1 = 0 \end{aligned}\right\} u(x) = a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\frac{\partial^2 u}{dx^2} = 2a_2 + 6a_3 x + 12 a_4 x^2$$

$$U = \frac{EI}{2}\int_0^{\ell} (2a_2 + 6a_3 x + 12 a_4 x^2)^2 dx$$

$$= \frac{2EI\ell}{5}\left[5a_2^2 + 5a_2 \ell(3a_3 + 4\ell a_4) + 3\ell^2(5a_3^2 + 15\ell a_3 a_4 + 12\ell^2 a_4^2)\right]$$

same as above

$$W = \int_0^{\ell} q(x) u(x)\, dx = q_0\left(\frac{\ell^3}{3}a_2 + \frac{\ell^4}{4}a_3 + \frac{\ell^5}{5}a_4\right)$$

$$\Pi = U - W$$

$$\delta\Pi = \delta(U - W) = \frac{\partial \Pi}{\partial a_2}\delta a_2 + \frac{\partial \Pi}{\partial a_3}\delta a_3 + \frac{\partial \Pi}{\partial a_4}\delta a_4 = 0$$

$$\left.\begin{aligned} \frac{\partial \Pi}{\partial a_2} &= 4\ell a_2 + 6\ell^2 a_3 + 8\ell^3 a_4 - \frac{q_0 \ell^3}{3} = 0 \\ \frac{\partial \Pi}{\partial a_3} &= 6\ell^2 a_2 + 12\ell^3 a_3 + 18\ell^4 a_4 - \frac{q_0 \ell^4}{4} = 0 \\ \frac{\partial \Pi}{\partial a_4} &= 8\ell^3 a_2 + 18\ell^4 a_3 + \frac{144\ell^5 a_4}{5} - \frac{q_0 \ell^5}{5} = 0 \end{aligned}\right\} a_2 = \frac{\ell^2 q}{4EI};\quad a_3 = -\frac{\ell q}{6EI};\quad a_4 = \frac{q}{24EI}$$

$$\therefore\ u(x) = \frac{qx^2}{24EI}(6\ell^2 - 4\ell x + x^2)$$

exact solution

TOPIC 2: Strong, Weak, and Galerkin Forms

The fundamental problem of the calculus of variations is to find the function $u(x)$ such that $\Pi(u) = \int_a^b F\left(x, u, \frac{du}{dx}\right)dx$ reaches an extreme value in the interval $x \in (a,b)$ with boundary conditions $u(a) = u_1$ and $u(b) = u_2$. F may include additional derivatives of u in x. The task of finding an extreme value for the functional $\Pi(u)$ is conceptually similar to finding the extrema of a function $f(x)$ defined as the x_i where $\left.\frac{\partial f}{\partial x}\right|_{x=x_i} = 0$; for $\Pi(u)$, the extrema results in $\delta\Pi = 0$.

$$\delta\Pi(u) = \delta\int_a^b F\left(x, u, \frac{du}{dx}\right)dx = \int_a^b \delta F\, dx$$

we ar varying the function, $u(x)$, not the variable x

$$= \int_a^b \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}(\delta u)'\right]dx$$

since $\delta\left(\frac{du}{dx}\right) = \frac{d}{dx}(\delta u)$ similar to $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for a continuous function, f.

$$= \int_a^b \frac{\partial F}{\partial u}\delta u\, dx + \cancelto{0}{\left.\frac{\partial F}{\partial u'}\delta u\right|_a^b} - \int_a^b \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\delta u\, dx$$

$\delta u = 0$ at $x = a$ and $x = b$ since it is known that $u(a) = u_1$ and $u(b) = u_2$.

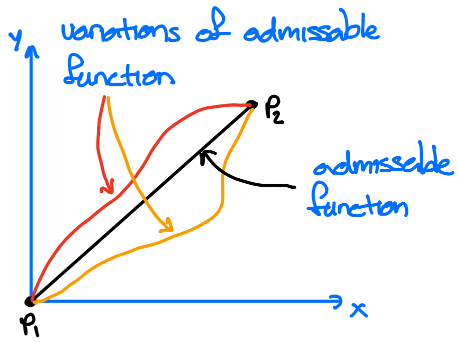
$$= \int_a^b \underbrace{\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]}_{=0}\delta u\, dx = 0$$

$$\int_a^b y\,dx = yx\Big|_a^b - \int_a^b x\,dy$$

$$\left.\begin{array}{l} \frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \\ \\ u(a) = u_1 \quad u(b) = u_2 \end{array}\right\}$$ strong form (S): must be satisfied at every point in the domain.

$$\int_a^b \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]\delta u\, dx = 0$$ weak form (W): must be satisfied over the interval $x \in [a,b]$, a less strict requirement than pointwise satisfaction. The FEM is based on this form.

Problem: find the shortest curve joining points P_1 $(x,y) = (0,0)$ and P_2 $(x,y) = (1,1)$.



differential curve length: $ds^2 = dx^2 + \left(\frac{dy}{dx}dx\right)^2$

$$\therefore \quad ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$$

$$\delta\Pi = \delta\int_{P_1}^{P_2}\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx \longrightarrow$$ find the curve, $y(x)$, that minimizes the curve length, s.

$$\left.\begin{array}{l} \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \\ \\ y(0) = 0 \quad y(1) = 1 \end{array}\right\}$$ strong form; $F = ds = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} dx$

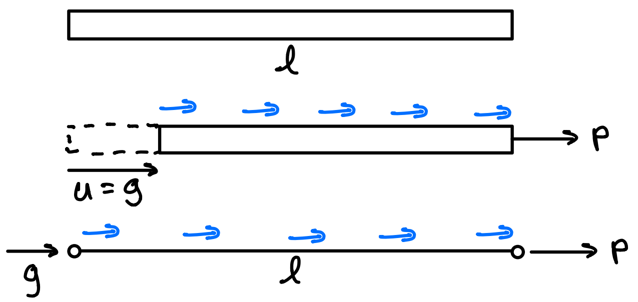
$$\frac{\partial}{\partial x}\left(\frac{y'}{\sqrt{1+(y')^2}}\right)=0 \qquad \therefore \quad \frac{y'}{\sqrt{1+(y')^2}}=c_0 \qquad \therefore \quad \frac{dy}{dx}=c_1$$

$$\int dy=\int c_1\,dx \longrightarrow y=c_1x+c_2$$

$$\left.\begin{aligned} c_1(0)+c_2&=0\\ c_1(1)+c_2&=1\end{aligned}\right\} \rightarrow \begin{bmatrix}0&1\\1&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix} \qquad \therefore \quad c_1=1,\ c_2=0$$

$y=x$ is the shortest curve joining P_1 and P_2.

Strong and Weak Forms for an Elastic Body



rod: supports axial loads only

$$\Pi=U-\left(\sum_{i=1}^{n}\bar{P}_i\right)\cdot u-\int_A\cancel{(\bar{T}dA)}^{\,0}\cdot u-\int_V(\bar{b}dV)\cdot u=0$$

$$U=\frac{1}{2}\int_V\sigma_x\varepsilon_x\,dA\,dx=\frac{1}{2E}\int_V\sigma_x^2\,dA\,dx=\frac{1}{2AE}\int_0^{\ell}F^2dx=\frac{AE}{2}\int_0^{\ell}\left(\frac{du}{dx}\right)^2dx$$

$$\varepsilon_x=\frac{\sigma_x}{E} \qquad \sigma_x=\frac{F}{A}=E\frac{du}{dx}$$

$$W=\int_V b_x u\,dA\,dx+Pu(\ell)=\int_0^{\ell}b_xA\,u\,dx+Pu(\ell)=\int_0^{\ell}fu\,dx+Pu(\ell)$$

$$\Pi=U-W=\int_0^{\ell}\left[\frac{AE}{2}\left(\frac{du}{dx}\right)^2-fu\right]dx-Pu(\ell)$$

$$\delta\Pi=\delta(U-W)=\int_0^{\ell}\left[AE\frac{du}{dx}\frac{\partial(\delta u)}{dx}-f\delta u\right]dx-P\delta u(\ell) \qquad AE\int_0^{\ell}\frac{du}{dx}\frac{\partial(\delta u)}{dx}dx=AE\frac{du}{dx}\delta u\Big|_0^{\ell}-AE\int_0^{\ell}\frac{\partial^2u}{dx^2}\delta u\,dx$$

$$=-\int_0^{\ell}(AEu_{xx}+f)\delta u\,dx+AE[u_x(\ell)\delta u(\ell)-u_x(0)\cancel{\delta u(0)}^{\,0}]-P\delta u(\ell) \qquad \text{since } \underline{u(0)=g}$$

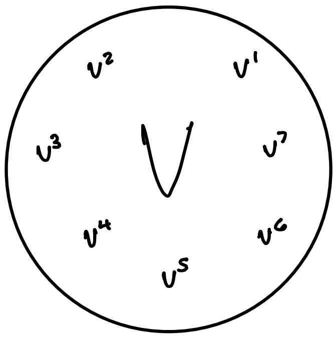
$$=-\int_0^{\ell}\underbrace{(AEu_{xx}+f)}_{=0}\delta u\,dx+\underbrace{[AEu_x(\ell)-P]}_{=0}\delta u(\ell)=0$$

$$\left.\begin{array}{lll} AEu_{xx}+f=0 & \text{governing equation} & \\ u(0)=g & \text{prescribed displacement} & \text{[Dirichlet (essential) BC]} \\ AEu_x(\ell)=P & \text{applied force} & \text{[Nuemann (natural) BC]}\end{array}\right\}(S)$$

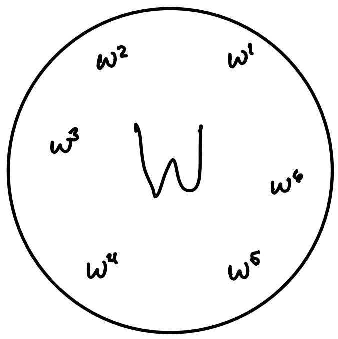
Requires continuity of the field variables in the domain and differentiability up to the order of the governing equation.

A boundary-value problem involves imposing boundary conditions on the field variable, u.

Although we could solve (S) directly, we will use this 1D problem to illustrate the FEM. Recall the Rayleigh-Ritz method had us define a trial solution and then minimize the total potential energy w.r.t. the solution parameters — we will essentially do the same thing for the FEM.



V is a collection of trial solutions which adhere to the kinematic constraints, here, $v(0) = g$.



W is a collection of weighting functions (virtual displacements) which vanish at the known kinematic constraints, $w(0) = 0$.

$w \equiv \delta u$

$$\int_0^\ell (EAu_{xx} + f)\, w\, dx = 0$$

$w \equiv \delta u$

$$EA\int_0^\ell u_{xx} w\, dx + \int_0^\ell f w\, dx = 0 \longleftarrow EA\int_0^\ell u_{xx} w\, dx = EAu_x w\Big|_0^\ell - EA\int_0^\ell u_x w_x\, dx$$

$$EA\int_0^\ell u_x w_x\, dx - \int_0^\ell f w\, dx - EAu_x w\Big|_0^\ell = 0$$

$$EA\int_0^\ell u_x w_x\, dx = \int_0^\ell f w\, dx + EA[u_x(\ell)w(\ell) - u_x(0)\cancelto{0}{w(0)}]$$

This integral relation is the weak (W) form, so named since integration by parts lowers the derivative requirement on the trial solution.

Apparently, the (S) solution u is also a (W) solution. We still do not know the solution u, but since the order of the derivative is lower in (W), we have an easier time finding a trial solution v, i.e., we need only consider trial solutions which possess first derivatives. But can such a trial solution of (W) be a solution of (S)?

$$EA\int_0^\ell v_x w_x\, dx = \int_0^\ell f w\, dx + Pw(\ell) \longleftarrow EA\int_0^\ell v_x w_x\, dx = EAv_x w\Big|_0^\ell - EA\int_0^\ell v_{xx} w\, dx$$

$$\int_0^\ell (EAv_{xx} + f) w\, dx + Pw(\ell) - EA[v_x(\ell)w(\ell) - v_x(0)\cancelto{0}{w(0)}] = 0$$

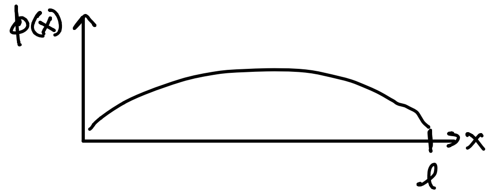
since any trial solution must have $v(0) = g$

$$\int_0^\ell (EAv_{xx} + f) w\, dx + w(\ell)[P - EAv_x(\ell)] = 0 \qquad [1]$$

$EAu_{xx} + f = 0$ and $EAu_x(\ell) = P$ must hold. Let's prove it!

Assume $w = \phi(EAu_{xx} + f)$ with $\phi(0) = \phi(\ell) = 0$. Remember, w may take any form so long as $w(0) = 0$ since the displacement is known at $x = 0$: $u(0) = g$. Thus, our assumption is perfectly legitimate.

Assume $\phi(x) \geq 0$



Substitute w into [1]:

$$\int_0^\ell (EAu_{xx} + f) w \, dx = \int_0^\ell (EAu_{xx} + f)^2 \phi \, dx = 0$$

since $w(\ell) = 0$

since $\phi > 0$ for $x \in (0, \ell)$ and $(EAu_{xx} + f)^2 \geq 0$, then $EAu_{xx} + f = 0$

Since $EAu_{xx} + f = 0$ is true for one choice of w, it is true for all legitimate forms of w. Now, assume an alternative form of w where $w(\ell) \neq 0$ [but $w(0) = 0$ is always true], then [1] becomes:

$w(\ell)[P - EAu_x(\ell)] = 0$ $\therefore$ $EAu_x(\ell) = P$ since $w(\ell) \neq 0$

The FEM is based upon an approximate version of (W) form:

$$EA \int_0^\ell u_x w_x \, dx = \int_0^\ell f w \, dx + Pw(\ell) \longrightarrow EA\,a(w, u) = (w, f) + Pw(\ell)$$

$a(w, u) = a(u, w)$ $\therefore$ symmetric stiffness matrix; force reciprocity

<u>Galerkin's Approximation</u>

1. Assume a form of $w \in W$ of the appropriate differential order (first order in our case)

   $w = \sum_{i=1}^{n} c_i N_i(x)$ c_i are tuning coefficients; N_i are basis, interpolation, or <u>shape functions</u>; $N_i(0) = 0$.

2. Use functions $w^* \in W$ to construct your trial solution $u \in U$.

   $u = w^* + \phi$ $w^* = \sum_{i=1}^{n} u_i N_i(x)$; $\phi = g N_{n+1}(x)$ $\therefore$ $N_{n+1}(0) = 1$

   since $u(0) = \cancel{w^*(0)}^{\,0} + \phi(0) = g$; notice that u and w are identical in construction up to ϕ!

3. Apply to (W):

$$EA\,a(w, w^* + \phi) = (w, f) + Pw(\ell) \longrightarrow EA\,a(w, w^*) = (w, f) + Pw(\ell) - EA\,a(w, \phi)$$

$$EA\,a\left[\sum_{i=1}^{n} c_i N_i(x), \sum_{j=1}^{n} u_j N_j(x)\right] = \left[\sum_{i=1}^{n} c_i N_i(x), f\right] + P\left[\sum_{i=1}^{n} c_i N(\ell)\right] - EA\,a\left[\sum_{i=1}^{n} c_i N_i(x), g N_{n+1}(x)\right]$$

$$\sum_{j=1}^{n} EA\,a\left[N_i(x), N_j(x)\right] u_j = \left[N_i(x), f\right] + PN_i(\ell) - EA\,a\left[N_i(x), gN_{n+1}(x)\right]$$

unknown displacement

c_i can be factored out because they appear every term.

Notice that the RHS contains all of our known quantities (g, P, N); the LHS contains the unknowns, u_i.

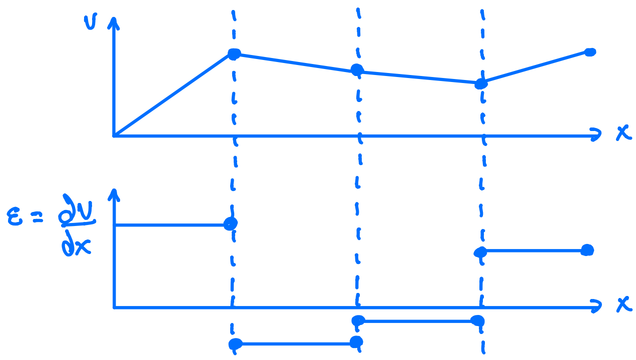
$$\underbrace{EA\int_0^{\ell}\begin{bmatrix} N_{1,x}N_{1,x} & N_{1,x}N_{2,x} & \cdots & N_{1,x}N_{n,x} \\ N_{2,x}N_{1,x} & N_{2,x}N_{2,x} & \cdots & N_{2,x}N_{n,x} \\ & & & \vdots \\ \text{symm.} & & & N_{n,x}N_{n,x} \end{bmatrix} dx}_{\substack{K \\ \text{stiffness matrix}}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}_{\substack{\bar{u} \\ \text{displacement} \\ \text{vector}}} = \underbrace{\int_0^{\ell}\begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix} f\,dx + P\begin{bmatrix} N_1(\ell) \\ N_2(\ell) \\ \vdots \\ N_n(\ell) \end{bmatrix} - gEA\int_0^{\ell} N_{n+1,x}\begin{bmatrix} N_{1,x} \\ N_{2,x} \\ \vdots \\ N_{n,x} \end{bmatrix} dx}_{\substack{\bar{f} \\ \text{force} \\ \text{vector}}}$$

Internal Loads
Nuemann BC
Dirichlet BC

What to consider when defining $N_i(x)$:

1. The number of N_i corresponds to the number of nodes. Out of a total of n N_i, m N_i are reserved for the m unknown u_i, i.e., $m = n - \#\text{constraints}$.

2. The N_i must be differentiable at least up to the order of v in (W).

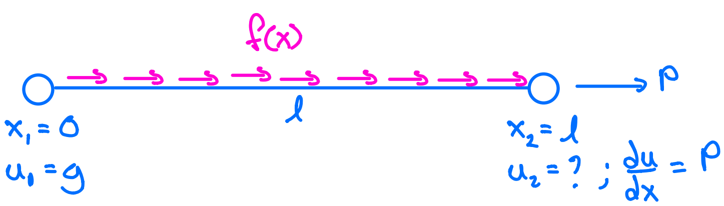


continuous displacements (C^0 continuity)

discontinuous strain (C^1 continuity violation)

3. $N_i(x_j) = \delta_{ij}$ so that u_i is not influenced by neighboring displacements.

Problem: Determine the displacement field, $u(x)$, over the rod by the Galerkin formulation.



This requires two shape functions (one for each node), $N_1(x)$ and $N_2(x)$: $N_1(x) = 1 - x/\ell$; $N_2(x) = x/\ell$

Notice, as basis functions, N_1 and N_2 span the whole linear function space (i.e., $c_1N_1 + c_2N_2$ gives any line):

$$y = c_1N_1 + c_2N_2 = c_1\left(1 - \frac{x}{\ell}\right) + c_2\frac{x}{\ell} = (c_2 - c_1)\frac{x}{\ell} + c_1 = mx + b \ \checkmark$$

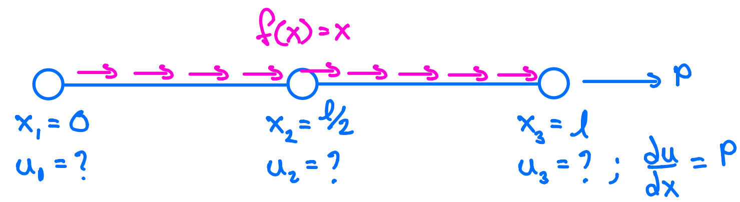
$$a(N_2,N_2)u_2 = (N_2,f) + PN_2(\ell) - a(N_2,N_1)g \qquad N_1 \equiv N_{n+1}$$

$$\underbrace{\left[\int_0^\ell \left(\frac{1}{\ell}\right)\left(\frac{1}{\ell}\right)dx\right]}_{K}\underbrace{u_2}_{\bar{u}} = \underbrace{\frac{1}{\ell}\int_0^\ell x f(x)dx + P - g\int_0^\ell \left(\frac{1}{\ell}\right)\left(-\frac{1}{\ell}\right)dx}_{f} \longrightarrow \frac{1}{\ell}u_2 = \frac{1}{\ell}\int_0^\ell x f(x)dx + P + \frac{g}{\ell}$$

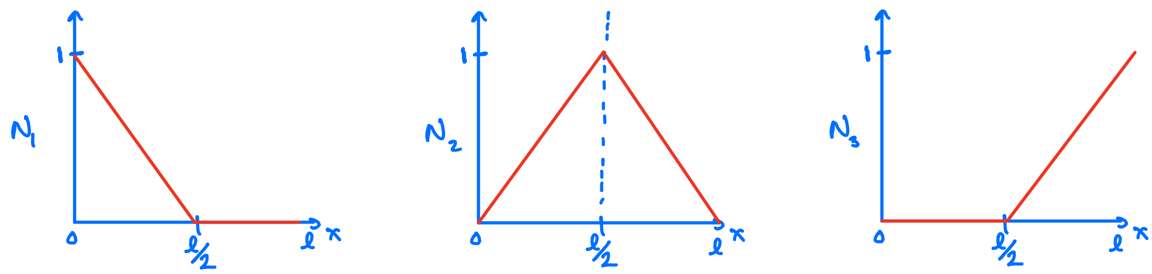
$$u(x) = u_1N_1 + u_2N_2 = \sum_i^n u_iN_i(x)$$

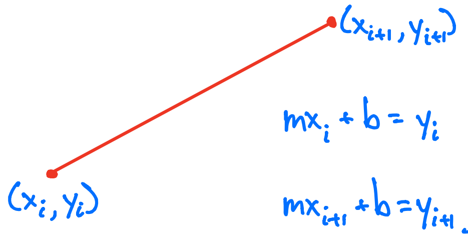
Notice that the rod has 2 d.o.f.s, but the problem has 1 constraint; therefore, we only need to solve for 1 d.o.f., i.e., u_2. This generally holds: # system d.o.f.s − # constraints = # unknowns.

Problem: Determine the displacement field, $u(x)$, over the rod by the Galerkin formulation.



This requires three shape functions:





$$\left.\begin{array}{l} mx_i + b = y_i \\ mx_{i+1} + b = y_{i+1} \end{array}\right\} \begin{bmatrix} x_i & 1 \\ x_{i+1} & 1 \end{bmatrix}\begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_i \\ y_{i+1} \end{bmatrix} \longrightarrow N(x) = x\left(\frac{y_{i+1}-y_i}{x_{i+1}-x_i}\right) - \frac{x_iy_{i+1} - x_{i+1}y_i}{x_{i+1}-x_i} \qquad x_i \le x \le x_{i+1}$$

$$N_1(x) = \begin{cases} -\frac{2x}{\ell} + 1 = \frac{\ell - 2x}{\ell} & 0 \le x \le \ell/2 \\ 0 & x \ge \ell/2 \end{cases} \qquad \frac{dN_1}{dx} = \begin{cases} -\frac{2}{\ell} & 0 \le x \le \ell/2 \\ 0 & x \ge \ell/2 \end{cases}$$

$$N_2(x) = \begin{cases} \frac{2x}{\ell} & 0 \le x \le \ell/2 \\ -\frac{2x}{\ell} + 2 = \frac{2(\ell - x)}{\ell} & \ell/2 \le x \le \ell \end{cases} \qquad \frac{dN_2}{dx} = \begin{cases} \frac{2}{\ell} & 0 \le x \le \ell/2 \\ -\frac{2}{\ell} & \ell/2 \le x \le \ell \end{cases}$$

$$N_3(x) = \begin{cases} 0 & x \le \ell/2 \\ \frac{2x}{\ell} - 1 = \frac{2x - \ell}{\ell} & \ell/2 \le x < \ell \end{cases} \qquad \frac{dN_3}{dx} = \begin{cases} 0 & 0 \le x \le \ell/2 \\ \frac{2}{\ell} & \ell/2 \le x < \ell \end{cases}$$

$$i \quad \sum_{j=1}^{3} a(N_i,N_j)u_j = (N_i,f) + PN_i(\ell)$$

$$\underbrace{\int_0^\ell \begin{bmatrix} N_{1,x}N_{1,x} & N_{1,x}N_{2,x} & N_{1,x}N_{3,x} \\ N_{2,x}N_{1,x} & N_{2,x}N_{2,x} & N_{2,x}N_{3,x} \\ N_{3,x}N_{1,x} & N_{3,x}N_{2,x} & N_{3,x}N_{3,x} \end{bmatrix} dx}_{K} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\bar{u}} = \underbrace{\int_0^\ell \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} f\,dx + p \begin{bmatrix} N_1(\ell) \\ N_2(\ell) \\ N_3(\ell) \end{bmatrix}}_{\bar{f}}$$

$$K_{11} = a(N_1,N_1) = \int_0^{\ell/2} \left(\frac{2}{\ell}\right)^2 dx = \frac{2}{\ell} \qquad K_{12} = K_{21} = a(N_1,N_2) = -\int_0^{\ell/2} \left(\frac{2}{\ell}\right)^2 dx = -\frac{2}{\ell} \qquad K_{13} = K_{31} = a(N_1,N_3) = 0$$

$$K_{22} = a(N_2,N_2) = \int_0^{\ell/2} \left(\frac{2}{\ell}\right)^2 dx + \int_{\ell/2}^{\ell} \left(\frac{2}{\ell}\right)^2 dx = \frac{4}{\ell} \qquad K_{23} = K_{32} = -\int_{\ell/2}^{\ell} \left(\frac{2}{\ell}\right)^2 dx = -\frac{2}{\ell}$$

$$K_{33} = a(N_3,N_3) = \int_{\ell/2}^{\ell} \left(\frac{2}{\ell}\right)^2 dx = \frac{2}{\ell}$$

$$K = \frac{2}{\ell} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

non-invertable: without a displacement BC, the problem is unsolvable because K is not invertable, without a displacement BC, the rod structure is "floating"; that is, $\Sigma F \neq 0$.

$$f_1 = \frac{1}{\ell}\int_0^{\ell/2} (\ell - 2x)x\,dx + N_1(\ell) = \frac{\ell^2}{24}$$

$$f_2 = \frac{2}{\ell}\int_0^{\ell/2} x^2\,dx + \frac{2}{\ell}\int_{\ell/2}^{\ell} (\ell - x)x\,dx + N_2(\ell) = \frac{\ell^2}{4}$$

$$f_3 = \frac{2}{\ell}\int_{\ell/2}^{\ell} (x-\ell)x\,dx + N_3(\ell) = \frac{5\ell^2}{24} + h$$

$$\bar{f} = \frac{\ell^2}{24}\begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix} + h \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$u(x) = \sum_{j=1}^{n} u_j N_j(x) = u_1N_1 + u_2N_2 + u_3N_3 + \cdots + u_nN_n = [N_1 \;\; N_2 \;\; N_3 \cdots N_n] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \bar{N}\bar{u}$$

$$\varepsilon(x) = \frac{du}{dx} = u_1N_{1,x} + u_2N_{2,x} + u_3N_{3,x} + \cdots + u_nN_{n,x} = [N_{1,x} \;\; N_{2,x} \;\; N_{3,x} \cdots N_{n,x}] \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \frac{\partial \bar{N}}{\partial x}\bar{u}$$

$$U = \frac{1}{2}\int_V \sigma(x)\varepsilon(x)\,dA\,dx = \frac{EA}{2}\int_0^\ell \varepsilon^2(x)\,dx = \frac{EA}{2}\int_0^\ell \bar{u}^T \frac{\partial \bar{N}}{\partial x}^T \frac{\partial \bar{N}}{\partial x}\bar{u}\,dx = \frac{\bar{u}^T}{2} \underbrace{EA\int_0^\ell \begin{bmatrix} N_{1,x}N_{1,x} & N_{1,x}N_{2,x} & N_{1,x}N_{3,x} \\ N_{2,x}N_{1,x} & N_{2,x}N_{2,x} & N_{2,x}N_{3,x} \\ N_{3,x}N_{1,x} & N_{3,x}N_{2,x} & N_{3,x}N_{3,x} \end{bmatrix} dx}_{K}\, \bar{u} = \frac{1}{2}\bar{u}^T K \bar{u}$$

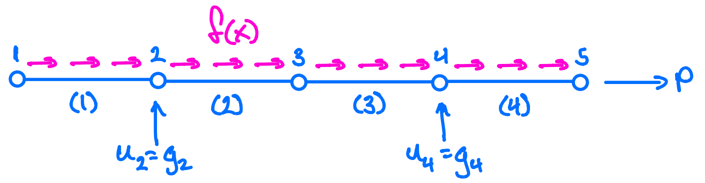
$\sigma(x) = E\varepsilon(x)$

$$W=\int_0^\ell f(x)u(x)\,dx + Pu(\ell) = \bar{u}^T\int_0^\ell \bar{N}^T f\,dx + \bar{u}^T\bar{N}^T(\ell)P = \bar{u}^T\underbrace{\left[\int_0^\ell \bar{N} f\,dx + \bar{N}^T(\ell)P\right]}_{\bar{f}} = \bar{u}^T\bar{f}$$

$$\Pi = U - W = \bar{u}^T\left(\tfrac{1}{2}K\bar{u}-\bar{f}\right)$$

$$\delta\Pi = \delta\bar{u}^T(K\bar{u}-\bar{f}) = 0 \qquad \delta\bar{u}^T \neq 0 \quad \therefore\ K\bar{u}=\bar{f}$$

Problem: Determine the displacement field, $u(x)$, over the rod by the Galerkin formulation.



unknown nodes, $\alpha = \{1,3,5\}$
known nodes, $\beta = \{2,4\}$

$$i \qquad \sum_j^{\alpha} EA\,a[N_i(x), N_j(x)]\,u_j = [N_i(x), f] + PN_i(\ell) - \sum_k^{\beta} EA\,a[N_i(x), gN_k(x)]$$

$$\underbrace{AE\int_0^\ell \begin{bmatrix} N_{1,x}N_{1,x} & N_{1,x}N_{3,x} & N_{1,x}N_{5,x} \\ & N_{3,x}N_{3,x} & N_{3,x}N_{5,x} \\ & & N_{5,x}N_{5,x} \end{bmatrix} dx}_{K}\ \underbrace{\begin{bmatrix} u_1 \\ u_3 \\ u_5 \end{bmatrix}}_{\bar{u}} = \underbrace{\int_0^\ell \begin{bmatrix} N_1 \\ N_3 \\ N_5 \end{bmatrix} f\,dx + P\begin{bmatrix} N_1(\ell) \\ N_3(\ell) \\ N_5(\ell) \end{bmatrix} - g_2 AE\int_0^\ell N_{2,x}\begin{bmatrix} N_{1,x} \\ N_{3,x} \\ N_{5,x} \end{bmatrix} dx - g_4 AE\int_0^\ell N_{4,x}\begin{bmatrix} N_{1,x} \\ N_{3,x} \\ N_{5,x} \end{bmatrix} dx}_{\bar{f}}$$

TOPIC 3:
The Element
Point of View

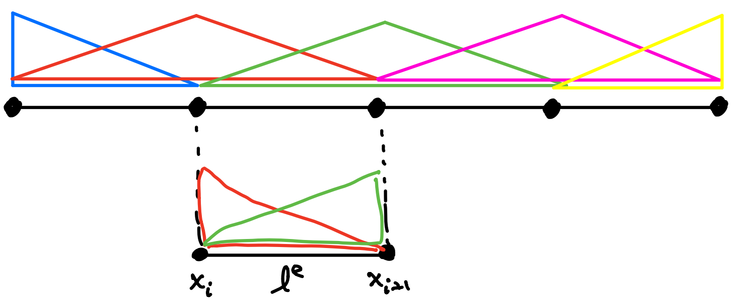
The governing equations of a typical problem are derived w.r.t. a coordinate system. So far, using the Galerkin formulation, we have carried the Cartesian coordinate system from the governing equation. This is very cumbersome in 1D with elements of varying length and gets worse in 2D/3D. Plus we must define a shape function for <u>every</u> node in the FE model!

Conversely, the Direct Stiffness Method (DSM) streamlines the FE procedure. The DSM (1) standrizes the formulation of elements (e.g., rod, quadrilateral, brick, etc.) (2) and then assembles a FE model out of a collection of elements. This procedure is much less cumbersome; much more convenient in practice. Task (1) is accomplished by utilizing an element coordinate in place of the Cartesian coordinate system.

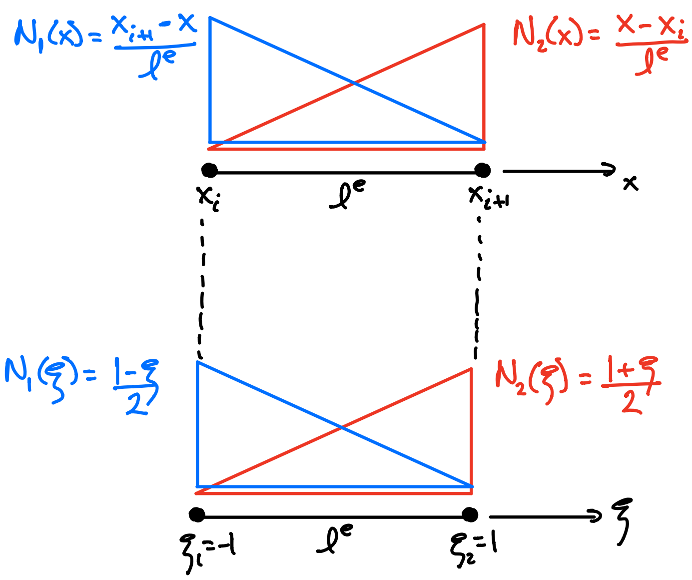
The Galerkin method defines and assembles the FE system of equations all in one step. The DSM splits the steps apart.

	Galerkin		DSM
Domain	$x \in [0, \ell]$		$\xi \in [-1, 1]$
Nodes	$x = x_i$	$i = 1 : \#\text{nodes}$	$\xi_1 = -1$; $\xi_2 = 1$
Shape Functions	$N_i(x)$	$i = 1 : \#\text{nodes}$	$N_1(\xi)$; $N_2(\xi)$

For 1D problem, but concept extends to higher dimensions



Mapping : x-ξ



Let $\xi(x) = c_1 + c_2 x$

$$\left.\begin{aligned} \xi(x_i) &= c_1 + c_2 x_i = \xi_1 = -1 \\ \xi(x_{i+1}) &= c_1 + c_2 x_{i+1} = \xi_2 = 1 \end{aligned}\right\} \begin{bmatrix} 1 & x_i \\ 1 & x_{i+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$c_1 = -\frac{x_i + x_{i+1}}{\ell^e} \quad ; \quad c_2 = \frac{2}{\ell^e}$$

$$\xi(x) = \frac{2x - x_i - x_{i+1}}{\ell^e} \quad \therefore \quad x(\xi) = \frac{\xi \ell^e + x_i + x_{i+1}}{2}$$

$$\left.\begin{aligned} N_1[x(\xi)] &= \frac{2x_{i+1} - \xi\ell^e - x_i - x_{i+1}}{2\ell^e} = \frac{1-\xi}{2} \\ N_2[x(\xi)] &= \frac{\xi\ell^e + x_i + x_{i+1} - 2x_i}{2\ell^e} = \frac{1+\xi}{2} \end{aligned}\right\}$$

Notice that $N_1 + N_2 = 1$. This ensures that interpolated displacements are not over/underrepresented. In general $\sum_i N_i = 1$.

Recall:

$$\sigma(x) = E\varepsilon(x) \qquad u(x) = N\bar{u} \qquad \frac{\partial N}{\partial x} = \frac{\partial N}{\partial \xi}\frac{\partial \xi}{\partial x} \qquad \frac{\partial \xi}{\partial x} = \frac{2}{\ell^e}$$

$$U^e = \frac{1}{2}\int_A\int_0^{\ell^e} \sigma(x)\varepsilon(x)\,dA\,dx = \frac{EA}{2}\int_0^{\ell^e} \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\,dx = \frac{EA}{2}\int_0^{\ell^e}\left(\frac{\partial N_i}{\partial x}\bar{u}\right)^T\left(\frac{\partial N_j}{\partial x}\bar{u}\right)dx$$

$$= \frac{1}{2}\bar{u}^T\frac{4EA}{[\ell^e]^2}\int_{-1}^{1}\frac{\partial N_i^T}{\partial \xi}\frac{\partial N_j}{\partial \xi}\left(\frac{\ell^e}{2}\right)d\xi\,\bar{u} = \frac{1}{2}\bar{u}^T\underbrace{\frac{2EA}{\ell^e}\int_{-1}^{1}\frac{\partial N_i^T}{\partial \xi}\frac{\partial N_j}{\partial \xi}\,d\xi}_{K^e}\,\bar{u} = \frac{1}{2}\bar{u}^T K^e \bar{u} \quad \text{strain energy}$$

K^e element stiffness matrix

$$K^e = \frac{2EA}{\ell^e}\int_{-1}^{1}\begin{bmatrix} N_{1,\xi}N_{1,\xi} & N_{1,\xi}N_{2,\xi} \\ N_{2,\xi}N_{1,\xi} & N_{2,\xi}N_{2,\xi}\end{bmatrix}d\xi = \frac{2EA}{\ell^e}\int_{-1}^{1}\begin{bmatrix} (-\frac{1}{2})(-\frac{1}{2}) & (-\frac{1}{2})(\frac{1}{2}) \\ (\frac{1}{2})(-\frac{1}{2}) & (\frac{1}{2})(\frac{1}{2})\end{bmatrix}d\xi$$

$$= \frac{EA}{2\ell^e}\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}\int_{-1}^{1}d\xi = \frac{EA}{\ell^e}\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}$$

Using element coordinates, we no longer need to know the element's x-position to determine K^e; we need only know its physical and geometric properties, i.e., E, A, and ℓ^e. The assembly process takes care of the position part.

$$W^e = \int_0^{\ell^e} f(x)u(x)\,dx = \bar{u}^T\int_0^{\ell} N_i(x)f(x)\,dx = \bar{u}^T\underbrace{\int_{-1}^{1} N_i(\xi)f(\xi)\frac{\ell^e}{2}\,d\xi}_{\bar{f}^e} = \bar{u}\bar{f}^e \quad \text{work done by distributed load}$$

$\bar{f}^e$ element force vector

The above formulation of $\bar{f}$ features $f(\xi)$ – a continuous function of ξ. Recall that $u(\xi) \approx \sum_j N_j(\xi)u_j$ where u_j are the unknown displacements at the nodes. Similarly, although we don't have too, we can approximate $f(\xi) \approx \sum_j N_j(\xi) f_j$ where $f_j = f(x_j)$. In that case:

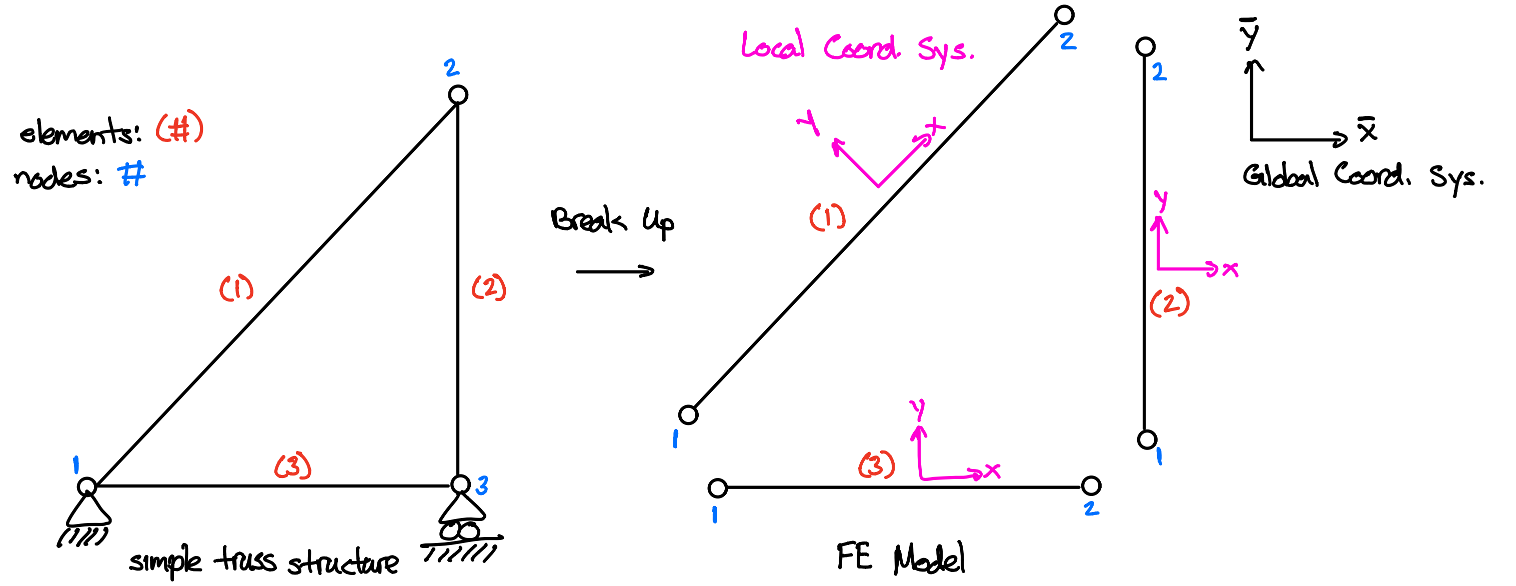
$$\bar{f}^e = \int_{-1}^{1} N_i(\xi)N_j(\xi)f_j\,d\xi \quad \text{element force vector (consistent)}$$

consistent just means that we used a shape function to approx. $f(x)$ just as we did for $u(x)$.

$$= \frac{\ell^e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2\end{bmatrix}\begin{bmatrix} f_1 \\ f_2\end{bmatrix}$$

these are not unknown: $f_1 = f(x_1)$ and $f_2 = f(x_2)$.

Notice that boundary conditions have not been mentioned. These are accounted for in the assembly process.



The rod FE matrix equations are based on the 1D governing equation $EA\frac{\partial^2 u}{\partial x^2} + f = 0$. In the local coordinate system, nodes displaces in either the x- or y-directions, but in the global coordinate system, the nodes may displace in both $\bar{X}$- and $\bar{Y}$-directions. All elements in a FE model must adhere to the same (global) coordinate system.

$$K^e = \frac{AE}{\ell^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} \xrightarrow[u_{1y} \text{ and } u_{2y}]{\text{expand to include}} \frac{AE}{\ell^e}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$$\bar{u}^e = \begin{bmatrix}u_{1x}\\ u_{2x}\end{bmatrix} \xrightarrow[u_{1y} \text{ and } u_{2y}]{\text{expand to include}} \begin{bmatrix}u_{1x}\\ u_{1y}\\ u_{2x}\\ u_{2y}\end{bmatrix} \qquad \bar{f}^e = \begin{bmatrix}f_{1x}\\ f_{2x}\end{bmatrix} \xrightarrow[f_{1y} \text{ and } f_{2y}]{\text{expand to include}} \begin{bmatrix}f_{1x}\\ f_{1y}\\ f_{2x}\\ f_{2y}\end{bmatrix}$$

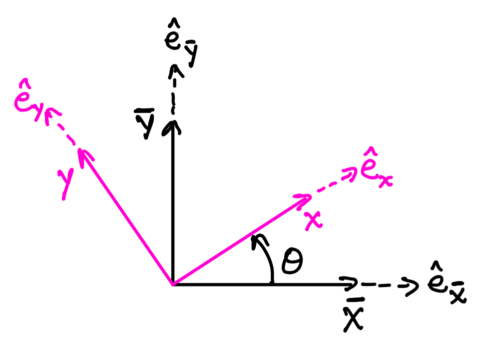
$$\bar{u}_1 = \begin{bmatrix}u_{1x}\\ u_{1y}\end{bmatrix} = \overbrace{u_{1x}\hat{e}_x + u_{1y}\hat{e}_y}^{\text{local}} = \overbrace{u_{1\bar{x}}\hat{e}_{\bar{x}} + u_{1\bar{y}}\hat{e}_{\bar{y}}}^{\text{global}}$$

$$\bar{u}_1 \cdot \hat{e}_x = u_{1x}\cancelto{1}{(\hat{e}_x \cdot \hat{e}_x)} + u_{1y}\cancelto{0}{(\hat{e}_y \cdot \hat{e}_x)} = u_{1x} = u_{1\bar{x}}(\hat{e}_{\bar{x}} \cdot \hat{e}_x) + u_{1\bar{y}}(\hat{e}_{\bar{y}} \cdot \hat{e}_x)$$

$$= u_{1\bar{x}}\cos\theta + u_{1\bar{y}}\sin\theta$$

$$\bar{u}_1 \cdot \hat{e}_y = u_{1x}\cancelto{0}{(\hat{e}_x \cdot \hat{e}_y)} + u_{1y}\cancelto{1}{(\hat{e}_y \cdot \hat{e}_y)} = u_{1y} = u_{1\bar{x}}(\hat{e}_{\bar{x}} \cdot \hat{e}_y) + u_{1\bar{y}}(\hat{e}_{\bar{y}} \cdot \hat{e}_y)$$

$$= -u_{1\bar{x}}\sin\theta + u_{1\bar{y}}\cos\theta$$



The same transformations can be done for $\bar{u}_2$ as well as be formulated to account for the z-coord. Together, the 2D local-global transformation is:

$$\overset{u^L}{\underbrace{\begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \end{bmatrix}}_{\text{local displacement vector}}} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix}}_{R \text{ rotation matrix}} \overset{u^G}{\underbrace{\begin{bmatrix} u_{1\bar{x}} \\ u_{1\bar{y}} \\ u_{2\bar{x}} \\ u_{2\bar{y}} \end{bmatrix}}_{\text{global displacement vector}}}$$

All along we've written K^e and $\bar{f}^e$ in terms of the local $\bar{u}^L$ displacements. We must move to the global system.

$$U = \frac{1}{2}(\bar{u}^L)^T K^e \bar{u}^L = \frac{1}{2}(R\bar{u}^G)^T K^e R\bar{u}^G = \frac{1}{2}(\bar{u}^G)^T \underbrace{R^T K^e R}_{K^e \text{ in global system}} \bar{u}^G = \frac{1}{2}(\bar{u}^G)^T K^e \bar{u}^G$$

$$K^e = \frac{E^e A^e}{l^e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix}$$

where $c = \cos\theta$; $s = \sin\theta$

Similarly: $\underset{\text{global}}{\bar{f}^e} = R^T \underset{\text{local}}{\bar{f}^L}$

TOPIC 4: Multi-freedom Constraints

Upon integrating DiffEqs of PDEs, we encounter integration constants which, depending on their value, alter the nature of the solution. In dynamics, the initial conditions, once known, identify a single solution out of the infinite family of solutions. In mechanics, the problem boundary conditions fulfill this role. In each case, ICs and BCs constrain the problem to identify a single solution.

We've already dealt with constraints in FEM on previous lectures as a matter of course. Now, let's have a more expanded treatment. The first three methods modify the unconstrained matrix equations after-the-fact; the forth method builds the matrix equations and accounts for the constraints simultaneously.

There are pros and cons to each method.

Leader-Follower Elimination and the Constraint Matrix: the DOFs involve in each constraint are separated into leader and follower DOFs. The follower DOFs are eliminated from the final matrix equations.

Penalty Augmentation: each constraint is enforced approximately by a fictious structural element (called penalty element). This involves augmenting the stiffness matrix to incorporate the stiffness (called weight, w) of this element. The constraint is approximate if $0 < w < \infty$; exactly if $w = \infty$ (but then the stiffness matrix becomes singular.

Lagrange Multiplier: for each constraint, an additional unknown/equation is adjoined to the matrix equations. these additional unknowns are, physically, the forces which enforce the constraints exactly.

Constraint Arrays: the constraints and their place in the final matrix equations are accounted for from the start and placed in arrays/tables.

Lead-Follower Elimination

We have applied SDOF constraints before, e.g., $u_{x4} = 0$ or $u_{y9} = 0.6$. Here, there is no "follower" DOF, nevertheless, the lead-follower method is just a generalization of what we've covered previously.

$$\overbrace{\begin{aligned} u_{x2} &= \tfrac{1}{2} u_{y2} \\ u_{x2} - 2u_{x4} + u_{x6} &= \tfrac{1}{4} \end{aligned}}^{\text{Multifreedom Constraints}} \quad \Bigg\} \quad \begin{bmatrix} 1 & -\tfrac{1}{2} \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{y2} \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} u_{x2} \\ u_{x4} \\ u_{x6} \end{bmatrix} = \frac{1}{4}$$

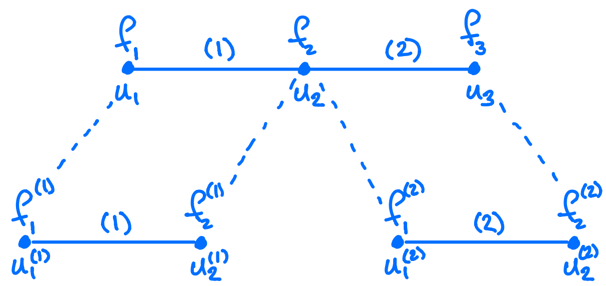
We don't consider nonlinear constraints in this class.

For this method, for each constraint equation, we would choose a single lead DOF (the choice is arbitrary) and then write the original vector of unknowns in terms of thes leaders: $u = Tu_L$. The matrix T is the constraint matrix.

$$u^F = Tu^L$$

following DOFs leading DOFs

Problem: Implement the following constraint on the system below.



$$K^{(e)} = \frac{A^{(e)}E^{(e)}}{l^{(e)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Step 1: Build (block diagonal) matrix equations without constraints.

$$K^0 = \begin{bmatrix} K^{(1)} & 0 \\ 0 & K^{(2)} \end{bmatrix} \qquad u^0 = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix} \qquad f^0 = \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{12}^{(1)} & k_{22}^{(1)} & 0 & 0 \\ 0 & 0 & k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & 0 & k_{12}^{(2)} & k_{22}^{(2)} \end{bmatrix}}_{K^0} \underbrace{\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}}_{u^0} = \underbrace{\begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_1^{(2)} \\ f_2^{(2)} \end{bmatrix}}_{f^0}$$

Step 2: Identify leaders and write original unknowns in terms of leaders.

$$\left.\begin{aligned} u_1^{(1)} &= u_1 \\ u_2^{(1)} &= u_2 \\ u_1^{(2)} &= u_2 \\ u_2^{(2)} &= u_3 \end{aligned}\right\} \rightarrow \underbrace{\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}}_{u^0} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{u^L}$$

Step 3: Ensure energetic consistency.

$$U = \frac{1}{2}(u^0)^T K^0 u^0 = \frac{1}{2}(u^L)^T T^T K^0 T u^L = \frac{1}{2}(u^L)^T K u^L$$

$$W = (u^0)^T f^0 = (u^L)^T T^T f^0 = (u^L)^T f$$

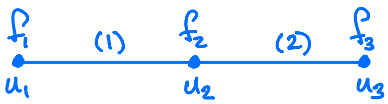
$$\delta\Pi = \delta(U - W) = \delta(u^L)^T (K u^L - f) = 0 \qquad \therefore\ K u^L = f$$

$$\begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 \\ k_{12}^{(1)} & k_{22}^{(1)}+k_{11}^{(2)} & k_{12}^{(2)} \\ 0 & k_{12}^{(2)} & k_{22}^{(2)} \end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)}+f_1^{(2)} \\ f_2^{(2)}\end{bmatrix}$$

Step 4: Solve $u^L = K^{-1} f$.

After $\bar{u}^L$ is known, the complete solution obtained via the transformation $\bar{u}^F = T\bar{u}^L$

Problem: Implement the following constraint on the system below.



$u_1 = g_1$

u_1 is a follower without a leader... one could consider g_1 as the "leader" even though it is a constant.

Step 1: Build matrix equations without constraints.

$$K^D = \begin{bmatrix} K^{(1)} & 0 \\ 0 & K^{(2)} \end{bmatrix} \qquad u^D = \begin{bmatrix} u^{(1)} \\ u^{(2)} \end{bmatrix} \qquad f^D = \begin{bmatrix} f^{(1)} \\ f^{(2)} \end{bmatrix}$$

Step 2: Identify leaders and write original unknowns in terms of leaders.

$$\left.\begin{aligned} u_1^{(1)} &= u_1 = g_1 \\ u_2^{(1)} &= u_2 \\ u_1^{(2)} &= u_2 \\ u_2^{(2)} &= u_3 \end{aligned}\right\} \longrightarrow \underbrace{\begin{bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \end{bmatrix}}_{u^D} = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} u_2 \\ u_3 \end{bmatrix}}_{u^L} + \underbrace{\begin{bmatrix} g_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{g}$$

Step 3: Ensure energetic consistancy.

$$U = \frac{1}{2}(u^D)^T K^D u^D = \frac{1}{2}[(u^L)^T T + g^T] K^D (T u^L + g) = \frac{1}{2}(u^L)^T T^T K^D T u^L + \frac{1}{2}(u^L)^T T^T K^D g + \frac{1}{2} g^T K^D T u^L + \frac{1}{2} g^T K^D g$$

$$W = (u^D)^T f^D = [(u^L)^T T^T + g^T] f^D = (u^L)^T T^T f^D + g^T f^D$$

$$\delta\Pi = \delta(U - W) = \delta(u^L)^T(\underbrace{T^T K^D T}_{K} u^L + \underbrace{T^T K^D}_{K_g} g - \underbrace{T^T f^D}_{f}) = 0$$

since $\delta\left[\frac{1}{2} g^T K g\right] = 0$ and $\delta[g^T f] = 0$

$$\therefore K u^L = f - K_g g$$

$$\begin{bmatrix} k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} \\ k_{12}^{(2)} & k_{22}^{(2)} \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \end{bmatrix} - g \begin{bmatrix} k_{12}^{(1)} \\ 0 \end{bmatrix}$$

For the case of followers without leaders, setting up the transformation equations is simple. The situation is barely more complicated if leaders are involved.

Problem: Implement the following constraint on the system below.

$u_2 + 2u_3 = g$ (multi-freedom constraint)

Step 2: either u_2 or u_3 may be chosen a leader.

$u_3 = -\frac{1}{2}u_2 + \frac{1}{2}g_3$ let $g = g_3$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g/2 \\ 0 \end{bmatrix}$$

Things become even more complicated if one constraint's follower is another's leader.

Problem: Implement the following constraint on the system below.

$u_2 - u_4 = 0; \quad u_2 + 2u_3 = g$

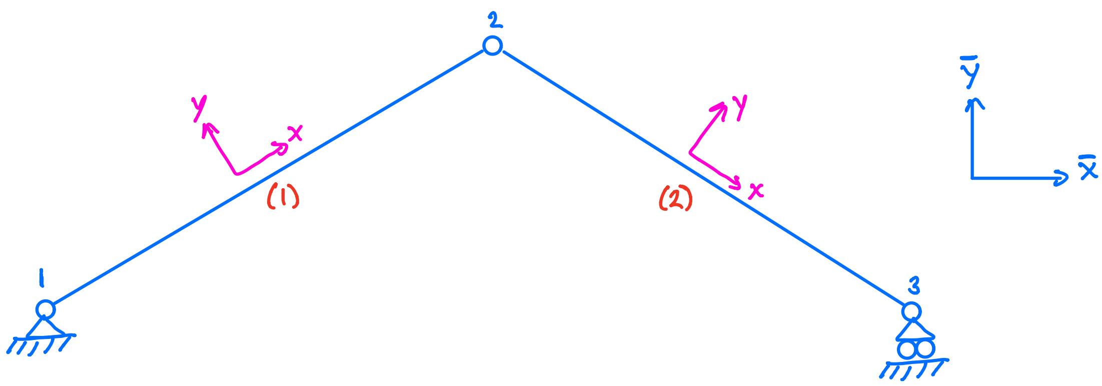
Step 2: If we write the constraints as $u_4 = u_2$ and $u_2 = -2u_3 + g$, then u_2 is both a leader and a follower. This is inconsistent. Instead, we find consistent expressions by choosing the common DOF, u_2, as leader.

$u_4 = u_2 \quad ; \quad u_3 = -\frac{1}{2}u_2 + \frac{1}{2}g$

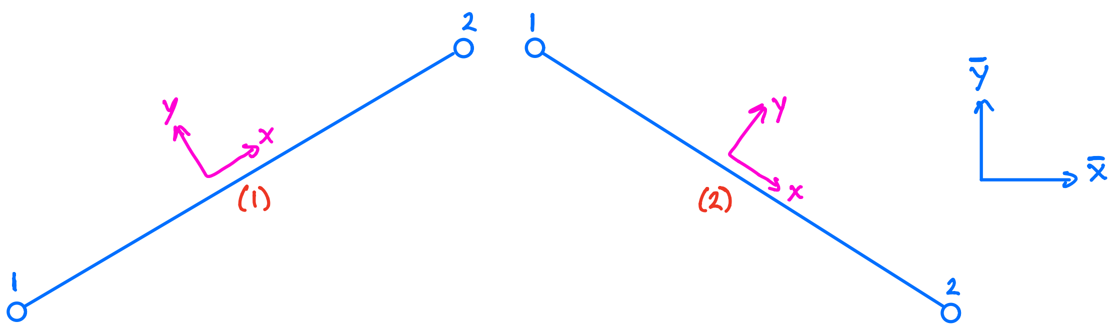
$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g/2 \\ 0 \end{bmatrix}$$

Notice that the size of our problem continues to decrease as we reduce the number of leaders and increase the number of followers.

Problem: Implement the following constraint on the system below.



Break-up



$$
\begin{bmatrix} u^{(1)}_{1x} \\ u^{(1)}_{1y} \\ u^{(1)}_{2x} \\ u^{(1)}_{2y} \\ u^{(2)}_{1x} \\ u^{(2)}_{1y} \\ u^{(2)}_{2x} \\ u^{(2)}_{2y} \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{3x} \end{bmatrix}
+
\begin{bmatrix} g_{1x} \\ g_{1y} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ g_{3y} \end{bmatrix}
$$

The method is exact up to computer precision.

The concept is easy to understand and implement.

PROs

For numerical stability, it is usually better to pick followers with larger coefficients. When the followers are eliminated, they take coefficients with them which staves off ill-conditioning K.

NEUTS

The choice of leaders and followers is difficult to program (i.e., requires user involvemen) and, thus, difficult to generalize.

Cannot be extended to nonlinear constraints.

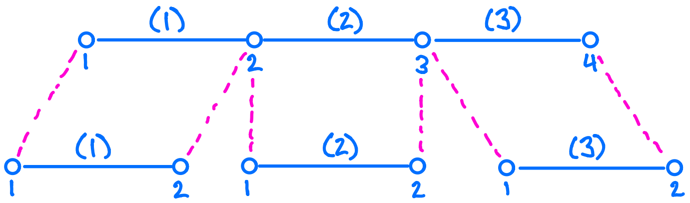
Computationally expensive: dedicates computer memory to "hold" the initial matrices, only to reduce them later.

CONS

Constraint Arrays

Each element is associated with a set of nodes; each node is associated with a number of degrees of freedom; each degree of freedom is represented by an equation in the FE system of equations. The constraint arrays reflect the element-node-DOF-equation association.

Problem: Determine the FE matrix equations for the 4-node (3-element) system using constraint arrays.



Local-Global (LG) Array

element (#) →	(1)	(2)	(3)
1	1	2	3
2	2	3	4

local node #

global node #

Equation Array (EA)

global node # →	1	2	3	4
x	1	2	3	4

DOF

equation #

From the EA, it is apparent that the global equations will have 4 eqns:

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \bar{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Element (1) is associated with global nodes 1 and 2 which correspond to global equations 1 and 2.

$$K^{(1)} = \frac{E^{(1)}A^{(1)}}{\ell^{(1)}} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \longleftrightarrow \overset{i}{\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} \overset{j}{\begin{Bmatrix} 1 & 2 \end{Bmatrix}} = \begin{Bmatrix} 11 & 12 \\ 21 & 22 \end{Bmatrix}$$

$$\bar{f}^{(1)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

$$K = K + K^{(1)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \bar{f} = \bar{f} + \bar{f}^{(1)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ 0 \\ 0 \end{bmatrix}$$

Element (2) is associated with global nodes 2 and 3 which correspond to global equations 2 and 3.

$$K^{(2)} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \begin{Bmatrix} 2 & 3 \end{Bmatrix} = \begin{Bmatrix} 22 & 23 \\ 32 & 33 \end{Bmatrix}$$

$$\bar{f}^{(2)} = \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$K = K + K^{(2)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \bar{f} = \bar{f} + \bar{f}^{(2)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} \\ 0 \end{bmatrix}$$

Element (3) is associated with global nodes 3 and 4 which correspond to global equations 3 and 4.

$$K^{(3)} = \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 3 \\ 4 \end{Bmatrix} \{3 \;\; 4\} = \begin{Bmatrix} 33 & 34 \\ 43 & 44 \end{Bmatrix} \qquad \bar{f}^{(3)} = \begin{bmatrix} f_1^{(3)} \\ f_2^{(3)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$$

$$K = K + K^{(3)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & 0 & 0 \\ k_{21}^{(1)} & k_{22}^{(1)} + k_{11}^{(2)} & k_{12}^{(2)} & 0 \\ 0 & k_{21}^{(2)} & k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \qquad \bar{f} = \bar{f} + \bar{f}^{(3)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_1^{(2)} \\ f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} \end{bmatrix}$$

Problem: Determine the FE matrix equations for the 4-node (3-element) system using constraint arrays.

$u_2 = d u_3$ ⟵ multifreedom constraints relate one DoF to other DoFs. We've already encounted simple constraints where a DoF has a fixed value, e.g., $u_2 = g$.

d: proportionality constant

Local-Global (LG) Array

element (#) ⟶	(1)	(2)	(3)
1	1	2	3
2	2	3	4

local node # (rows); global node # (entries)

Since u_2 and u_3 are related by $u_2 = d u_3$, only one of these is treated as unknown: once we solve for either u_2 or u_3, we automatically know the other.

In this case (one DoF proportional to another) we choose either u_2 or u_3 as unknown, thus we could follow two EAs:

u_3 is unknown *

global node # ⟶	1	2	3	4
x	1	2d	2	3

DOF x; equation # (entries); $u_2 = d u_3$

u_2 is unknown

global node # ⟶	1	2	3	4
x	1	2	2 1/d	3

DOF x; equation # (entries); $u_3 = \frac{1}{d} u_2$

In the EA, the unknow DoFs get an equation number; the dependent DoFs acquire the same equation number multiplied by a proportionality constant.

From the EA, it is apparent that the global equations will have 3 eqns:

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \bar{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} u_1 \\ u_3 \\ u_4 \end{bmatrix}$$

Element (1) is associated with global nodes 1 and 2 which correspond to global equations 1 and 2.

$$K^{(1)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 1 \\ 2\alpha \end{Bmatrix} \begin{Bmatrix} 1 & 2\alpha \end{Bmatrix} = \begin{Bmatrix} 11 & 12\alpha \\ 21\alpha & 22\alpha^2 \end{Bmatrix} \qquad \bar{f}^{(1)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 1 \\ 2\alpha \end{Bmatrix}$$

$$K = K + K^{(1)} = \begin{bmatrix} k_{11}^{(1)} & \alpha k_{12}^{(1)} & 0 \\ \alpha k_{21}^{(1)} & \alpha^2 k_{22}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \bar{f} = \bar{f} + \bar{f}^{(1)} = \begin{bmatrix} f_1^{(1)} \\ \alpha f_2^{(1)} \\ 0 \end{bmatrix}$$

Element (2) is associated with global nodes 2 and 3 which correspond to global equations 2α and 2.

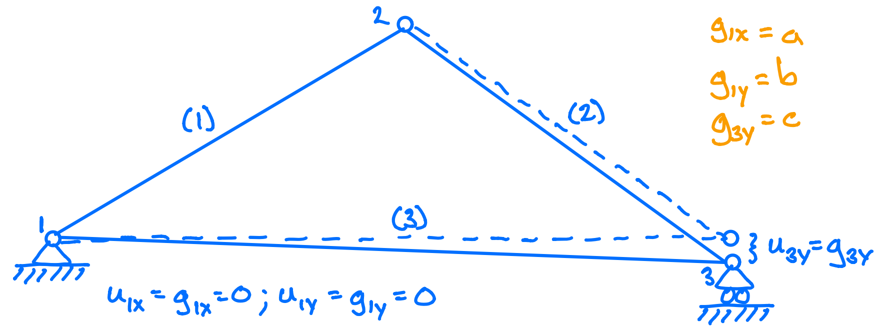
$$K^{(2)} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 2\alpha \\ 2 \end{Bmatrix} \begin{Bmatrix} 2\alpha & 2 \end{Bmatrix} = \begin{Bmatrix} 22\alpha^2 & 22\alpha \\ 22\alpha & 22 \end{Bmatrix} \qquad \bar{f}^{(2)} = \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 2\alpha \\ 2 \end{Bmatrix}$$

$$K = K + K^{(2)} = \begin{bmatrix} k_{11}^{(1)} & \alpha k_{12}^{(1)} & 0 \\ \alpha k_{21}^{(1)} & \alpha^2 (k_{22}^{(1)} + k_{11}^{(2)}) + \alpha (k_{12}^{(2)} + k_{21}^{(2)}) + k_{22}^{(2)} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \bar{f} = \bar{f} + \bar{f}^{(2)} = \begin{bmatrix} f_1^{(1)} \\ \alpha (f_2^{(1)} + f_1^{(2)}) + f_2^{(2)} \\ 0 \end{bmatrix}$$

Element (3) is associated with global nodes 3 and 4 which correspond to global equations 2 and 3.

$$K^{(3)} = \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 2 \\ 3 \end{Bmatrix} \begin{Bmatrix} 2 & 3 \end{Bmatrix} = \begin{Bmatrix} 22 & 23 \\ 32 & 33 \end{Bmatrix} \qquad \bar{f}^{(3)} = \begin{bmatrix} f_1^{(3)} \\ f_2^{(3)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 2 \\ 3 \end{Bmatrix}$$

$$K = K + K^{(3)} = \begin{bmatrix} k_{11}^{(1)} & \alpha k_{12}^{(1)} & 0 \\ \alpha k_{21}^{(1)} & \alpha^2 (k_{22}^{(1)} + k_{11}^{(2)}) + \alpha (k_{12}^{(2)} + k_{21}^{(2)}) + k_{22}^{(2)} + k_{11}^{(3)} & k_{12}^{(3)} \\ 0 & k_{21}^{(3)} & k_{22}^{(3)} \end{bmatrix} \qquad \bar{f} = \bar{f} + \bar{f}^{(3)} = \begin{bmatrix} f_1^{(1)} \\ \alpha (f_2^{(1)} + f_1^{(2)}) + f_2^{(2)} + f_1^{(3)} \\ f_2^{(3)} \end{bmatrix}$$



$g_{1x} = a$
$g_{1y} = b$
$g_{3y} = c$

Problem: Determine the FE matrix equations using constraint arrays.

Local-Global (LG) Array

element (#) →

local node #	(1)	(2)	(3)
1	1	2	1
2	2	3	3

$\underbrace{\qquad\qquad}_{\text{global node \#}}$

Equation Array (EA)

global node # →

DOF	1	2	3
x	a	1	3
y	b	2	c

$\underbrace{\qquad\qquad}_{\text{equation \#}}$

In the EA, prescribed displacements recieve no equation number.

From the EA, it is apparent that the global equations will have 3 eqns:

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \bar{f} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \bar{u} = \begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{3x} \end{bmatrix}$$

Element (1) is associated with global nodes 1 and 2 which correspond to global equations 1 and 2.

$$K^{(1)} = \begin{bmatrix} k_{11}^{(1)} & k_{12}^{(1)} & k_{13}^{(1)} & k_{14}^{(1)} \\ k_{21}^{(1)} & k_{22}^{(1)} & k_{23}^{(1)} & k_{24}^{(1)} \\ k_{31}^{(1)} & k_{32}^{(1)} & k_{33}^{(1)} & k_{34}^{(1)} \\ k_{41}^{(1)} & k_{42}^{(1)} & k_{43}^{(1)} & k_{44}^{(1)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} a \\ b \\ 1 \\ 2 \end{Bmatrix} \{a \;\; b \;\; 1 \;\; 2\} \begin{matrix} \times \\ = \times \\ \\ \\ \end{matrix} \begin{bmatrix} a^2 & ab & 1a & 2a \\ ab & b^2 & 1b & 2b \\ 1a & 1b & 11 & 12 \\ 2a & 2b & 21 & 22 \end{bmatrix}$$

$$\bar{f}^{(1)} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \\ f_4^{(1)} \end{bmatrix} \longleftrightarrow \begin{matrix} \times \\ \times \\ \\ \\ \end{matrix} \begin{Bmatrix} a \\ b \\ 1 \\ 2 \end{Bmatrix}$$

$$K = K + K^{(1)} = \begin{bmatrix} k_{33}^{(1)} & k_{34}^{(1)} & 0 \\ k_{43}^{(1)} & k_{44}^{(1)} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{f} = \bar{f} + \bar{f}^{(1)} - \bar{f}_g^{(1)} = \begin{bmatrix} f_3^{(1)} - k_{31}^{(1)} g_{1x} - k_{32}^{(1)} g_{1y} \\ f_4^{(1)} - k_{41}^{(1)} g_{1x} - k_{42}^{(1)} g_{1y} \\ 0 \end{bmatrix}$$

Element (2) is associated with global nodes 2 and 3 which correspond to global equations 1, 2, and 3.

$$K^{(2)} = \begin{bmatrix} k_{11}^{(2)} & k_{12}^{(2)} & k_{13}^{(2)} & k_{14}^{(2)} \\ k_{21}^{(2)} & k_{22}^{(2)} & k_{23}^{(2)} & k_{24}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} & k_{34}^{(2)} \\ k_{41}^{(2)} & k_{42}^{(2)} & k_{43}^{(2)} & k_{44}^{(2)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} 1 \\ 2 \\ 3 \\ c \end{Bmatrix} \{1 \;\; 2 \;\; 3 \;\; c\} = \begin{matrix} \\ \\ \\ \times \end{matrix} \begin{bmatrix} 11 & 12 & 13 & 1c \\ 21 & 22 & 23 & 2c \\ 31 & 32 & 33 & 3c \\ 1c & 2c & 3c & c^2 \end{bmatrix}$$

$$\bar{f}^{(2)} = \begin{bmatrix} f_1^{(2)} \\ f_2^{(2)} \\ f_3^{(2)} \\ f_4^{(2)} \end{bmatrix} \longleftrightarrow \begin{matrix} \\ \\ \\ \times \end{matrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \\ c \end{Bmatrix}$$

$$K = K + K^{(2)} = \begin{bmatrix} k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{13}^{(2)} \\ k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & k_{23}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} \end{bmatrix}$$

$$\bar{f} = \bar{f} + \bar{f}^{(2)} - \bar{f}_g^{(2)} = \begin{bmatrix} f_3^{(1)} + f_1^{(2)} - k_{31}^{(1)} g_{1x} - k_{32}^{(1)} g_{1y} - k_{14}^{(2)} g_{3y} \\ f_4^{(1)} + f_2^{(2)} - k_{41}^{(1)} g_{1x} - k_{42}^{(1)} g_{1y} - k_{24}^{(2)} g_{3y} \\ f_3^{(2)} - k_{34}^{(2)} g_{3y} \end{bmatrix}$$

Element (3) is associated with global nodes 2 and 3 which correspond to global equations 1, 2, and 3.

$$K^{(3)} = \begin{bmatrix} k_{11}^{(3)} & k_{12}^{(3)} & k_{13}^{(3)} & k_{14}^{(3)} \\ k_{21}^{(3)} & k_{22}^{(3)} & k_{23}^{(3)} & k_{24}^{(3)} \\ k_{31}^{(3)} & k_{32}^{(3)} & k_{33}^{(3)} & k_{34}^{(3)} \\ k_{41}^{(3)} & k_{42}^{(3)} & k_{43}^{(3)} & k_{44}^{(3)} \end{bmatrix} \longleftrightarrow \begin{Bmatrix} a \\ b \\ 3 \\ c \end{Bmatrix} \{a \;\; b \;\; 3 \;\; c\} = \begin{matrix} \times \\ \times \\ \\ \times \end{matrix} \begin{Bmatrix} a^2 & ab & 3a & ac \\ ab & b^2 & 3b & bc \\ 3a & 3b & 33 & 3c \\ ac & bc & 3c & c^2 \end{Bmatrix}$$

$$\bar{f}^{(3)} = \begin{bmatrix} f_1^{(3)} \\ f_2^{(3)} \\ f_3^{(3)} \\ f_4^{(3)} \end{bmatrix} \longleftrightarrow \begin{matrix} \times \\ \times \\ \\ \times \end{matrix} \begin{Bmatrix} a \\ b \\ 3 \\ c \end{Bmatrix}$$

$$K = K + K^{(3)} = \begin{bmatrix} k_{33}^{(1)} + k_{11}^{(2)} & k_{34}^{(1)} + k_{12}^{(2)} & k_{13}^{(2)} \\ k_{43}^{(1)} + k_{21}^{(2)} & k_{44}^{(1)} + k_{22}^{(2)} & k_{23}^{(2)} \\ k_{31}^{(2)} & k_{32}^{(2)} & k_{33}^{(2)} + k_{33}^{(3)} \end{bmatrix}$$

$$\bar{f} = \bar{f} + \bar{f}^{(3)} - \bar{f}_g^{(3)} = \begin{bmatrix} f_3^{(1)} + f_1^{(2)} - k_{31}^{(1)} g_{1x} - k_{32}^{(1)} g_{1y} - k_{14}^{(2)} g_{3y} \\ f_4^{(1)} + f_2^{(2)} - k_{41}^{(1)} g_{1x} - k_{42}^{(1)} g_{1y} - k_{24}^{(2)} g_{3y} \\ f_3^{(2)} + f_3^{(3)} - k_{31}^{(3)} g_{1x} - k_{32}^{(3)} g_{1y} - (k_{34}^{(2)} + k_{34}^{(3)}) g_{3y} \end{bmatrix}$$

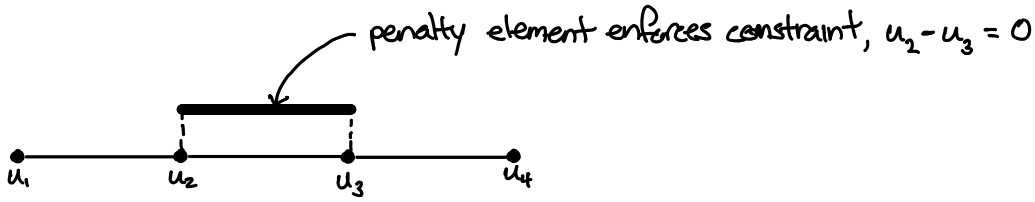
$$= \begin{bmatrix} f_3^{(1)} + f_1^{(2)} - k_{14}^{(2)} g_{3y} \\ f_4^{(1)} + f_2^{(2)} - k_{24}^{(2)} g_{3y} \\ f_3^{(2)} + f_3^{(3)} - (k_{34}^{(2)} + k_{34}^{(3)}) g_{3y} \end{bmatrix} \quad \text{since } g_{1x} = g_{1y} = 0$$

The method of constraint arrays has the same PROs, CONs, and NEUTs issues as the Lead-Follower method. There is, however, one additional advantage: constraint arrays reveal the final size of the FE matrix equations which is computationally efficient; the lead-follower method starts with the maximum number of matrix equations and then reduces them (computationally expensive in reguard to computer memory allocation).

Penalty Augmentation Method

The penalty method augments the stiffness matrix and force vector to incorporate an imaginary, near rigid elastic element. The specific stiffness of this imaginary element is the penalty weight.

Consider the system below:



$$u_2 = u_4 \longrightarrow u_2 - u_4 = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} u_2 \\ u_4 \end{bmatrix}}_{\tilde{u}} = 0$$

$$U = \frac{1}{2} u^T \underbrace{C^T C}_{K^p} u = \frac{1}{2} u^T K^p u$$

$$W = u^T \underbrace{C^T (0)}_{f^p} = u^T f^p$$

$$\underbrace{\overset{K^p}{\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \overset{f^p}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

looks like a conventional stiffness matrix for one element!

$$\omega \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ matrix equations for fictitious element

penalty weight (larger ω means more rigid; smaller ω means more flexible)

$$\underbrace{\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22}+\omega & k_{23}-\omega & 0 \\ 0 & k_{32}-\omega & k_{33}+\omega & k_{34} \\ 0 & 0 & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}}_{\text{augmented matrix equations}} \longleftarrow \omega \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

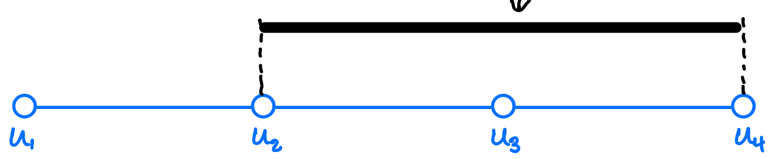
From here, we solve as usual: $\bar{u} = K^{-1} \bar{f}$.

Since ω is finite (i.e., the penalty element isn't perfecty rigid), the constraint is only approximately satisfied, i.e., $u_2 - u_3 = \varepsilon \neq 0$. The magnitude of the constraint violation $|\varepsilon|$ depends on ω; the larger ω, the smaller the violation. It also makes K ill-conditioned with respect to inversion.

Square-root Rule: suppose the largest coefficient (in magnitude) in K is of the order 10^p and your computer precision is of order 10^q, then choose ω on the order of $10^p\sqrt{10^q}$.

Problem: Consider the system and constraint below. Enact the constraint using the penalty method.

penalty element (added to original system to enforce constraint)



$u_2 - u_4 = g$

$$u_2 - u_4 = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{C} \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = g$$

$$\omega \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = g \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

in this case, $g \neq 0$ leads to a penalty force in addition to the penalty stiffness.

$$\underbrace{\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22}+\omega & k_{23} & -\omega \\ 0 & k_{32} & k_{33} & k_{34} \\ 0 & -\omega & k_{43} & k_{44}+\omega \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + g \\ f_3 \\ f_4 - g \end{bmatrix}}_{\text{augmented matrix equations}} \longleftarrow \omega \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_2 \\ u_4 \end{bmatrix} = g \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Problem: Consider the system and constraint below. Enact the constraint using the penalty method.

$u_1 - 2u_2 + 3u_4 = 0$

can't draw this penalty element, but it's the method that's important.

$$u_1 - 2u_2 + u_4 = \underbrace{\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}}_{C} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix}$$

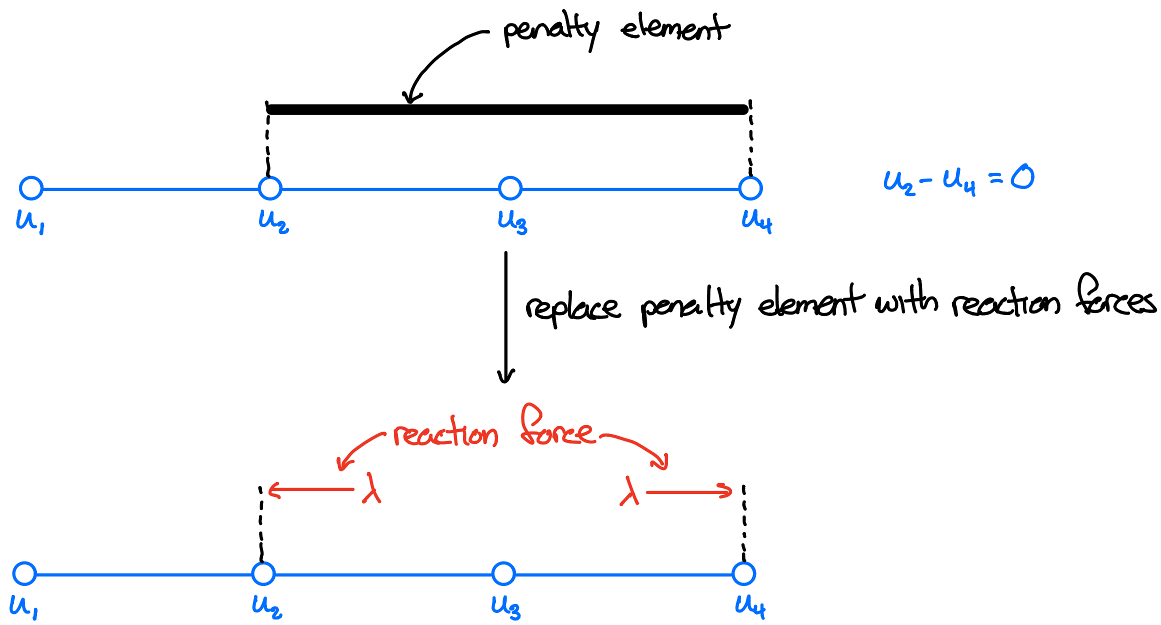
$$\omega \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} k_{11}+\omega & k_{12}-2\omega & 0 & 3\omega \\ k_{21}-2\omega & k_{22}+4\omega & k_{23} & -6\omega \\ 0 & k_{32} & k_{33} & k_{34} \\ 3\omega & -6\omega & k_{43} & k_{44}+9\omega \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}}_{\text{augmented matrix equations}} \longleftarrow \omega \underbrace{\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

this appears to be an odd stiffness matrix – some off-diagonal entries are positive! The concept of a penalty element is just a convenient visual representation of the method.

Lagrange Multiplier Adjunction

The Lagrange multiplier method is similar to the penalty method, except that the penalty element is replaced by unknown reaction forces that guarantee the constraint. It has its basis in variational calculus, but we'll skip the derivation and go straight to illustrating the procedure.



Augment the original matrix equations to include the reaction forces (Lagrange multiplier), λ.

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 \\ 0 & k_{32} & k_{33} & k_{34} \\ 0 & 0 & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 - \lambda \\ f_3 \\ f_4 + \lambda \end{bmatrix}$$

Since λ is unknown, transfer it to the vector of unknowns and augment the stiffness matrix.

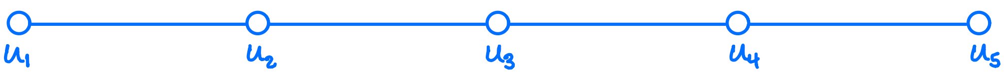
$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & 1 \\ 0 & k_{32} & k_{33} & k_{34} & 0 \\ 0 & 0 & k_{43} & k_{44} & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

To render the system of equations determinate, append the constraint equation to the stiffness matrix.

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & 1 \\ 0 & k_{32} & k_{33} & k_{34} & 0 \\ 0 & 0 & k_{43} & k_{44} & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ 0 \end{bmatrix} \quad \longleftarrow u_2 - u_4 = 0$$

The equations can now be solved for the unknown displacements and unknown reaction force.

Now, we demonstrate the general procedure using an example problem.



$u_2 - u_4 = g; \quad u_1 - 2u_2 + 3u_4 = 0$

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & 0 \\ 0 & k_{32} & k_{33} & k_{34} & 0 \\ 0 & 0 & k_{43} & k_{44} & k_{45} \\ 0 & 0 & 0 & k_{54} & k_{55} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

Step 1: Append the constraint equations to the matrix equations.

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & 0 \\ 0 & k_{32} & k_{33} & k_{34} & 0 \\ 0 & 0 & k_{43} & k_{44} & k_{45} \\ 0 & 0 & 0 & k_{54} & k_{55} \\ 0 & 1 & 0 & -1 & 0 \\ 1 & -2 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ g \\ 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ \\ \\ \longleftarrow u_2 - u_4 = g \\ \longleftarrow u_1 - 2u_2 + 3u_4 = 0 \end{array}$$

Step 2: Append a reaction force to the vector of unknowns — one for each constraint.

$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 \\ k_{21} & k_{22} & k_{23} & 0 & 0 \\ 0 & k_{32} & k_{33} & k_{34} & 0 \\ 0 & 0 & k_{43} & k_{44} & k_{45} \\ 0 & 0 & 0 & k_{54} & k_{55} \\ 0 & 1 & 0 & -1 & 0 \\ 1 & -2 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ g \\ 0 \end{bmatrix}$$

Step 3: Render the system of equations determinate by appending the transpose of the appended rows of the stiffness matrix.

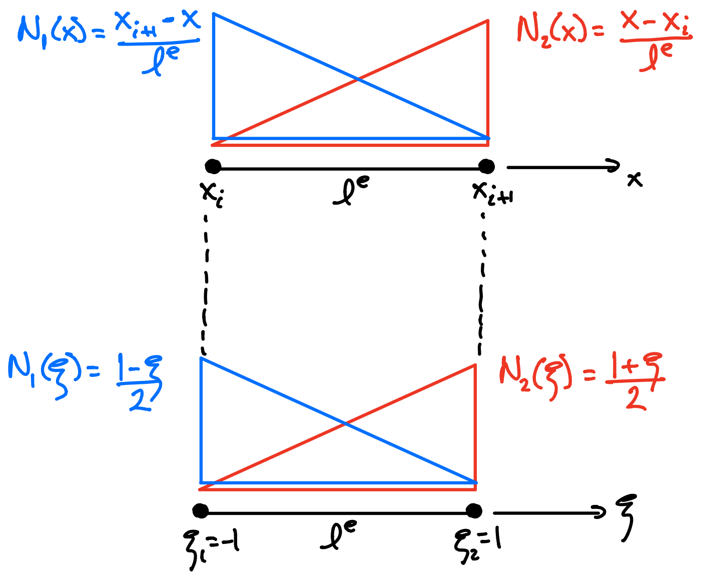
$$\begin{bmatrix} k_{11} & k_{12} & 0 & 0 & 0 & 0 & 1 \\ k_{21} & k_{22} & k_{23} & 0 & 0 & 1 & -2 \\ 0 & k_{32} & k_{33} & k_{34} & 0 & 0 & 0 \\ 0 & 0 & k_{43} & k_{44} & k_{45} & -1 & 3 \\ 0 & 0 & 0 & k_{54} & k_{55} & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ g \\ 0 \end{bmatrix}$$

The equations can now be solved for the unknown displacements and unknown reaction force.

Easy to implement and has excellent accuracy } PROS

It increases the size of the matrix equations with increasingly sparse rows/columns which is computationally undesirable in regard to memory allocation. } CONS

TOPIC 5:
Three-node Triangle



Let $\xi(x) = c_1 + c_2 x$

$$\left.\begin{aligned} \xi(x_i) &= c_1 + c_2 x_i = \xi_1 = -1 \\ \xi(x_{i+1}) &= c_1 + c_2 x_{i+1} = \xi_2 = 1 \end{aligned}\right\} \begin{bmatrix} 1 & x_i \\ 1 & x_{i+1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$c_1 = -\frac{x_i + x_{i+1}}{\ell^e} \; ; \; c_2 = \frac{2}{\ell^e}$$

$$\xi(x) = \frac{2x - x_i - x_{i+1}}{\ell^e} \quad \therefore \; x(\xi) = \frac{\xi \ell^e + x_i + x_{i+1}}{2}$$

$$U^e = \frac{1}{2}\int_0^{\ell^e}\int_A \sigma(x)\varepsilon(x)\,dA\,dx = \frac{1}{2}\bar{u}^T \underbrace{EA\int_0^{\ell^e} \frac{\partial \bar{N}^T}{\partial x}\frac{\partial \bar{N}}{\partial x}\,dx}_{K^e}\,\bar{u} \longleftarrow \frac{\partial \bar{N}}{\partial x} = \frac{\partial N}{\partial \xi}\frac{d\xi}{dx} = \frac{2}{\ell^e}\frac{\partial N}{\partial \xi} \quad \text{chain rule}$$

$$dx = \frac{\ell^e}{2}\,d\xi$$

$$K^e = \frac{2EA}{\ell^e}\int_{-1}^{1} \frac{\partial \bar{N}^T}{\partial \xi}\frac{\partial \bar{N}}{\partial \xi}\,d\xi = \frac{EA}{\ell^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

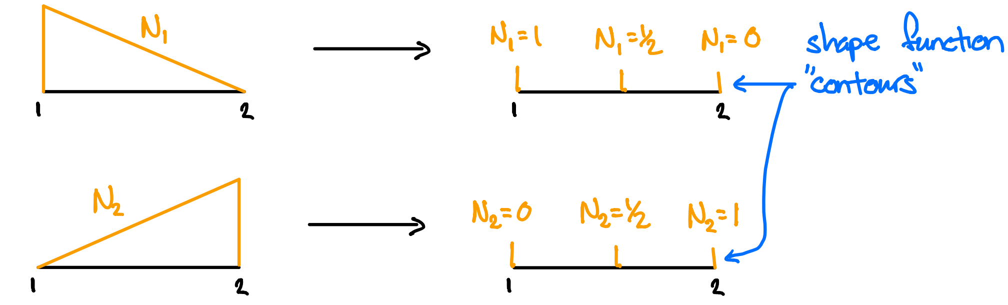
Using element coordinates, we no longer need to know the element's x-position to determine K^e; we need only know its physical and geometric properties, i.e., E, A, and ℓ^e. The assembly process takes care of the position part.

$$W^e = \int_0^{\ell^e} f(x)u(x)\,dx = \bar{u}^T \underbrace{\int_0^{\ell^e} \bar{N}^T f\,dx}_{\bar{f}^e} \longleftarrow dx = \frac{\ell^e}{2}\,d\xi$$

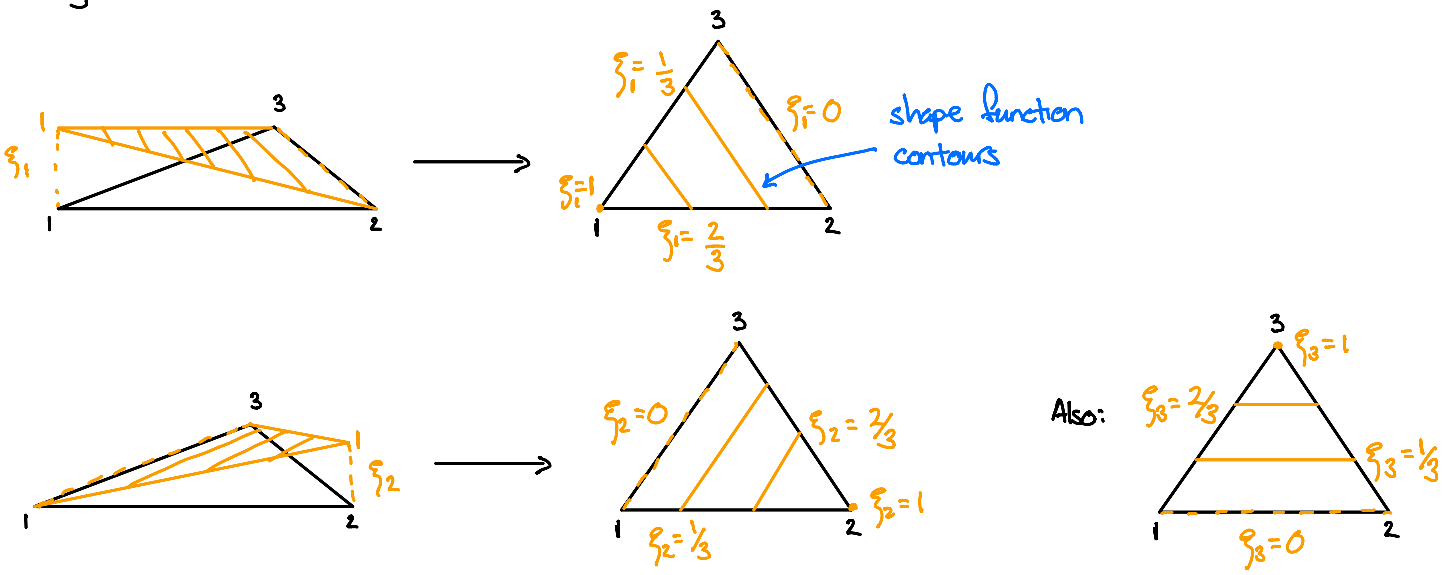
$$\bar{f}^e = \int_{-1}^{1} \bar{N}^T f \frac{\ell^e}{2}\,d\xi \longleftarrow f = \bar{N}\bar{f} \qquad \text{á la } u = \bar{N}\bar{u}$$

$$= \frac{\ell^e}{2}\int_{-1}^{1} \bar{N}^T \bar{N} \bar{f}\,d\xi \quad \text{consistent force vector}$$

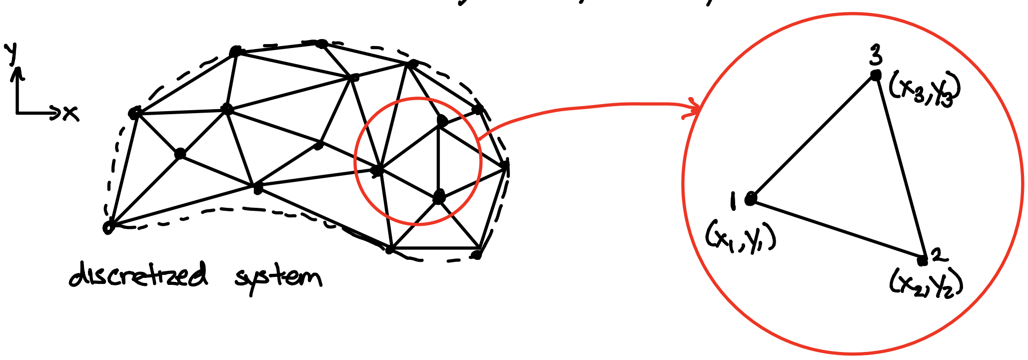
Rod Element



Triangular Element



Triangular Shape Functions: $N_1 = \xi_1$, $N_2 = \xi_2$, $N_3 = \xi_3$

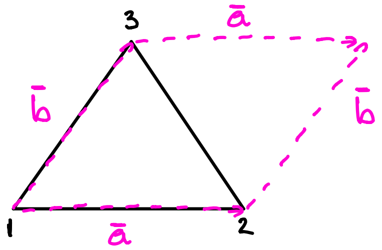


$\xi(x,y) = c_0 + c_2 x + c_3 y$ mapping from Cartesian to Triangular coordinates

$$\left.\begin{aligned} \xi_1 = c_0^1 + c_1^1 x_1 + c_2^1 y_1 &= 1 \\ c_0^1 + c_1 x_2 + c_2^1 y_2 &= 0 \\ c_0^1 + c_1^1 x_3 + c_2^1 y_3 &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} c_0^1 \\ c_1^1 \\ c_2^1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \therefore \; \xi_1 = \overbrace{\frac{x_2 y_3 - x_3 y_2}{2A}}^{c_0^1} + \overbrace{\frac{y_2 - y_3}{2A}}^{c_1^1} x + \overbrace{\frac{x_3 - x_2}{2A}}^{c_2^1} y$$

$$\left.\begin{aligned} \xi_2 = c_0^2 + c_1^2 x_1 + c_2^2 y_1 &= 0 \\ c_0^2 + c_1^2 x_2 + c_2^2 y_2 &= 1 \\ c_0^2 + c_1^2 x_3 + c_2^2 y_3 &= 0 \end{aligned}\right\} \rightarrow \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} c_0^2 \\ c_1^2 \\ c_2^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \therefore \; \xi_2 = \overbrace{\frac{x_3 y_1 - x_1 y_3}{2A}}^{c_0^2} + \overbrace{\frac{y_3 - y_1}{2A}}^{c_1^2} x + \overbrace{\frac{x_1 - x_3}{2A}}^{c_2^2} y$$

$$\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix} = \underset{\uparrow\ \text{triangle area}}{\frac{1}{2A}}\begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}\begin{bmatrix}1\\ x\\ y\end{bmatrix} \quad \therefore \quad \begin{bmatrix}1\\ x\\ y\end{bmatrix} = \begin{bmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\\ \xi_3\end{bmatrix}$$



The triangle area is half the area of a parallelogram with the triangle forming one corner.

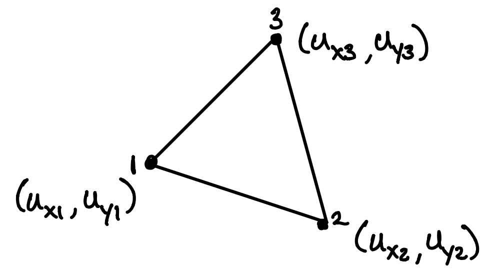
$$2A = |\bar{a} \times \bar{b}| = \begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}$$

$$U^e = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_0^{h^e}\int_A \underbrace{\bar{\varepsilon}^T C}_{\sigma}\bar{\varepsilon}\, dA\, dz$$

$$W^e = \int_V f\bar{s}\, dU = \int_0^{h^e}\int_A f\bar{s}\, dA\, dz$$

to avoid ambiguity, let the displacement function $\bar{u}(x,y) \longrightarrow \bar{s}(x,y)$ and the displacement vector $\bar{u} = [u_{1x}\ u_{1y}\ \cdots\ u_{nx}\ u_{ny}]^T$.

$$\bar{s}(x,y) = \begin{bmatrix}u_x(x,y)\\ u_y(x,y)\end{bmatrix} = N\bar{u} = \begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0\\ 0 & N_1 & 0 & N_2 & 0 & N_3\end{bmatrix}\begin{bmatrix}u_{x1}\\ u_{y1}\\ u_{x2}\\ u_{y2}\\ u_{x3}\\ u_{y3}\end{bmatrix}$$



$$\bar{\varepsilon}(x,y) = \begin{bmatrix}\varepsilon_x(x,y)\\ \varepsilon_y(x,y)\\ \varepsilon_{xy}(x,y)\end{bmatrix} = D\bar{s} = \begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}N\bar{u} = \underbrace{\begin{bmatrix}N_{1,x} & 0 & N_{2,x} & 0 & N_{3,x} & 0\\ 0 & N_{1,y} & 0 & N_{2,y} & 0 & N_{3,y}\\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & N_{3,y} & N_{3,x}\end{bmatrix}}_{B}\bar{u} = B\bar{u}$$

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi_i}\frac{\partial \xi_i}{\partial x} \ ; \quad \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi_i}\frac{\partial \xi_i}{\partial y}$$

In this case, $\frac{\partial N_i}{\partial \xi_i} = 1$; $\frac{\partial \xi_i}{\partial x}$ and $\frac{\partial \xi_i}{\partial y}$ are determined from the above coordinate mapping

$$B = \frac{1}{2A}\begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0\\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1\\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2\end{bmatrix}$$

$$U^e = \frac{1}{2}\int_V \sigma\varepsilon\, dV = \frac{1}{2}\int_0^{h^e}\int_A \underbrace{\bar{\varepsilon}^T C}_{\sigma}\bar{\varepsilon}\, dA dz = \frac{1}{2}\bar{u}^T \underbrace{h^e A^e B^T C B}_{K^e}\bar{u}$$

$$W^e = \int_V f u\, dV = \bar{u}^T \int_0^{h^e}\int_A N^T f\, dA dz$$

note: $\int_A \xi_1^i \xi_2^j \xi_3^k \, dA = \frac{i!j!k!}{(2+i+j+k)!}A$ where $i,j,k \geq 0$

Validation of Shape Functions

continuity: along any triangle side, the variation of $u_x(x,y)$ and $u_y(x,y)$ is uniquely determined by the displacement of the nodes on that size only.

$$u_x = \sum_i^3 N_i u_{x_i} = u_{x1}\xi_1 + u_{x2}\xi_2 + u_{x3}\xi_3$$

on side 1-2: $u_x = u_{x1}\xi_1 + u_{x2}\xi_2$ since $\xi_3 = 0$

on side 1-3: $u_x = u_{x1}\xi_1 + u_{x3}\xi_3$ since $\xi_2 = 0$

on side 1-2: $u_x = u_{x2}\xi + u_{x3}\xi$ since $\xi_1 = 0$

same for $u_y = \sum_i^3 N_i u_{y_i}$

continuity satisfied

completeness: the shape functions are capable of describing a general linear displacement field.

$$u_x = a_0 + a_1 x + a_2 y$$

$$\begin{aligned} u_x = \sum_i^3 N_i u_{x_i} &= u_{x1}\xi_1 + u_{x2}\xi_2 + u_{x3}\xi_3 \\ &= (a_0 + a_1 x_1 + a_2 y_1)\xi_1 + (a_0 + a_1 x_2 + a_2 y_2)\xi_2 + (a_0 + a_1 x_3 + a_2 y_3)\xi_3 \\ &= a_0 \sum_i^3 \xi_i + a_1 \sum_i^3 x_i \xi_i + a_2 \sum_i^3 y_i \xi_i \\ &= a_0 + a_1 x + a_2 y \end{aligned}$$

completeness satisfied

same for $u_y = \sum_i^3 N_i u_{y_i}$

TOPIC 6:
Four-node Quadrilateral

Partial derivatives of shape functions w.r.t. Cartesian coordinates are required for stress/strain calculations and, ultimately, K^e and f^e. These derivatives are non-trivial when the shape functions are not directly functions of x and y, but of element coordinates.

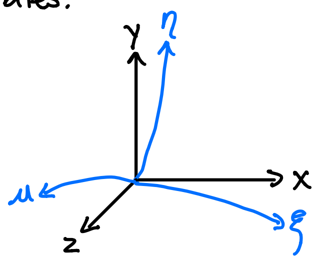
$$U^e = \int_V \varepsilon^T C \varepsilon \, dV \ ; \quad W^e = \int_V f u \, dV$$

derivatives

The Jacobian relates differentials among coordinates:

$$y = f(x) \qquad\qquad x = \frac{\xi \ell^e + x_i + x_{i+1}}{2}$$

$$dy = \underbrace{f'(x)}_{J} dx \qquad\qquad dx = \underbrace{\frac{\ell^e}{2}}_{J} d\xi$$



$$x(\xi,\eta,u) \ ; \ y(\xi,\eta,u) ; \ z(\xi,\eta,u)$$

$$\left.\begin{aligned} dx &= \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta + \frac{\partial x}{\partial u} du \\ dy &= \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta + \frac{\partial y}{\partial u} du \\ dz &= \frac{\partial z}{\partial \xi} d\xi + \frac{\partial z}{\partial \eta} d\eta + \frac{\partial z}{\partial u} du \end{aligned}\right\} \longrightarrow \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} & \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \eta} & \frac{\partial z}{\partial u} \end{bmatrix}}_{J^T} \begin{bmatrix} d\xi \\ d\eta \\ du \end{bmatrix}$$

$$\therefore \ J = \frac{\partial(x,y,z)}{\partial(\xi,\eta,u)} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{bmatrix}$$

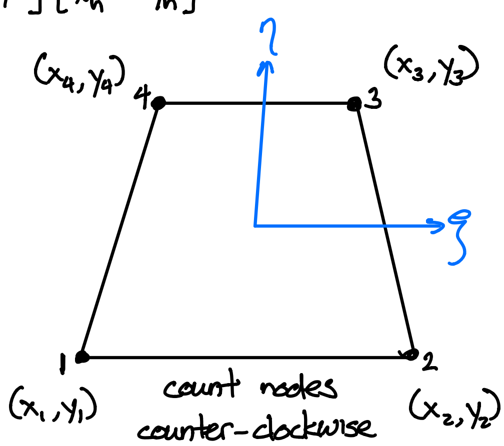
Jacobian matrix of (x,y,z) w.r.t. (ξ,η,u).

$$x = \sum_{i=1}^{n} x_i N_i(\xi,\eta); \quad y = \sum_{i=1}^{n} y_i N_i(\xi,\eta);$$

$$\left.\begin{aligned} \frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} x_i \frac{\partial N_i}{\partial \xi} \qquad \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} y_i \frac{\partial N_i}{\partial \xi} \end{aligned}\right\} \longrightarrow J = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

same for $\frac{\partial}{\partial \eta}$

$$\left.\begin{aligned} \frac{\partial N_i}{\partial x} &= \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_i}{\partial y} &= \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y} \end{aligned}\right\} \Rightarrow \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{J^{-1}} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$



$K^e = h^e \int_A B^T C B \, dA$ where (in 2D) $B = \begin{bmatrix} N_{1,x} & 0 & N_{2,x} & 0 & \cdots & N_{n,x} & 0 \\ 0 & N_{1,y} & 0 & N_{2,y} & \cdots & 0 & N_{n,y} \\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & \cdots & N_{n,y} & N_{n,x} \end{bmatrix}$

Numerical integration is routine in FEM, especially for iso-P elements. Standard practice is to use Gauss integration which uses a user-specified number of evaluation points to evaluate an integral to a given accuracy.

In 1D: $\int_{-1}^{1} F(\phi)\, d\phi \approx \sum_{i=1}^{p} \omega_i F(\phi_i)$

p: number of evaluation points $(p \geq 1)$
ω_i: weights
$\phi_i \in [-1, 1]$

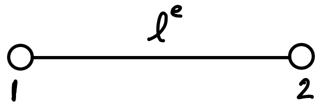
Rule of Thumb: p points integrates polynomials up to order q. In general, $2p-1 \geq q$.

Table 17.1 - One-Dimensional Gauss Rules with 1 through 5 Sample Points

Points	Rule
1	$\int_{-1}^{1} F(\xi)\, d\xi \approx 2F(0)$
2	$\int_{-1}^{1} F(\xi)\, d\xi \approx F(-1/\sqrt{3}) + F(1/\sqrt{3})$
3	$\int_{-1}^{1} F(\xi)\, d\xi \approx \frac{5}{9}F(-\sqrt{3/5}) + \frac{8}{9}F(0) + \frac{5}{9}F(\sqrt{3/5})$
4	$\int_{-1}^{1} F(\xi)\, d\xi \approx w_{14}F(\xi_{14}) + w_{24}F(\xi_{24}) + w_{34}F(\xi_{34}) + w_{44}F(\xi_{44})$
5	$\int_{-1}^{1} F(\xi)\, d\xi \approx w_{15}F(\xi_{15}) + w_{25}F(\xi_{25}) + w_{35}F(\xi_{35}) + w_{45}F(\xi_{45}) + w_{55}F(\xi_{55})$

For the 4-point rule, $\xi_{34} = -\xi_{24} = \sqrt{(3 - 2\sqrt{6/5})/7}$, $\xi_{44} = -\xi_{14} = \sqrt{(3 + 2\sqrt{6/5})/7}$, $w_{14} = w_{44} = \frac{1}{2} - \frac{1}{6}\sqrt{5/6}$, and $w_{24} = w_{34} = \frac{1}{2} + \frac{1}{6}\sqrt{5/6}$.
For the 5-point rule, $\xi_{55} = -\xi_{15} = \frac{1}{3}\sqrt{5 + 2\sqrt{10/7}}$, $\xi_{45} = -\xi_{35} = \frac{1}{3}\sqrt{5 - 2\sqrt{10/7}}$, $\xi_{35} = 0$, $w_{15} = w_{55} = (322 - 13\sqrt{70})/900$, $w_{25} = w_{45} = (322 + 13\sqrt{70})/900$ and $w_{35} = 512/900$.

Problem: determine the stiffness matrix of the 2-node rod element below using Gauss integration.



$N_1 = \frac{1}{2}(1-\xi)$ $\quad N_2 = \frac{1}{2}(1+\xi)$

$x = x_1 N_1 + x_2 N_2$

$dx = x_1 \frac{\partial N_1}{\partial \xi} d\xi + x_2 \frac{\partial N}{\partial \xi} d\xi = \frac{x_2 - x_1}{2} d\xi = \frac{\ell^e}{2} d\xi$

$u(x) = [N_1 \ \ N_2]\bar{u} = N\bar{u}$ $\quad \therefore \quad$ $\varepsilon(x) = DN\bar{u} = B\bar{u}$

differential operator, $D = \frac{d}{dx}$

$$B=\underbrace{\begin{bmatrix}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x}\end{bmatrix}}_{\text{strain-displacement matrix}}=\frac{1}{\ell^e}\begin{bmatrix}-1 & 1\end{bmatrix}$$

since $\frac{\partial N_1}{\partial x}=\frac{\partial N_1}{\partial \xi}\frac{\partial \xi}{\partial x}=\left(-\frac{1}{2}\right)\left(\frac{2}{\ell^e}\right)$ $\quad \frac{\partial N_2}{\partial x}=\frac{\partial N_2}{\partial \xi}\frac{\partial \xi}{\partial x}=\left(\frac{1}{2}\right)\left(\frac{2}{\ell^e}\right)$

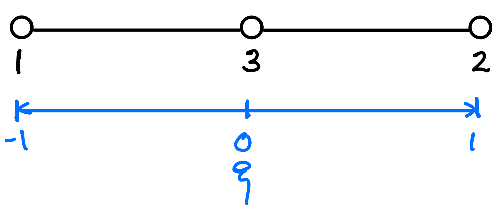
$$K^e=AE\int_0^{\ell^e}B^TB\,dx=\frac{AE\ell^e}{2}\int_{-1}^{1}B^TB\,d\xi=\frac{AE}{2\ell^e}\int_{-1}^{1}\underbrace{\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}}_{F(\xi)}d\xi$$

The polynomial degree is zero, and so we use $p=1$ integration points.

$$K^e\approx\frac{AE}{2\ell^e}\begin{bmatrix}2F(0) & 2F(0)\\ 2F(0) & 2F(0)\end{bmatrix}=\frac{AE}{\ell^e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

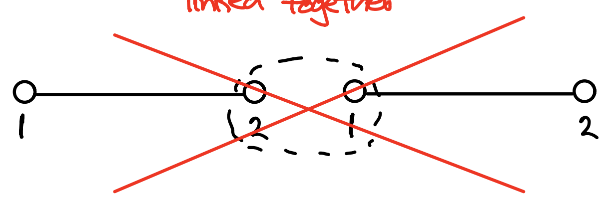
In general, we may have an integral of $F(\phi)$ over the domain $\phi\in[a,b]$. For Gauss integration we must have the integral over $\xi\in[-1,1]$. To this end, we use the linear mapping $\phi=\frac{a}{2}(1-\xi)+\frac{b}{2}(1+\xi)$, therefore, $\xi=\frac{2}{b-a}\left[\phi-\frac{a+b}{2}\right]$. Consequently $d\phi=\frac{1}{2}(b-a)d\xi$.

Problem: determine the stiffness matrix of the 3-node rod element below using Gauss integration.

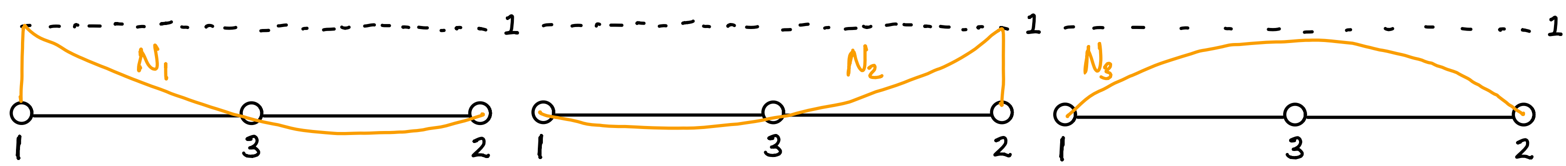


3-Node Rod

NOT two, 2-Node rods linked together



$N_1=\frac{1}{2}\xi(\xi-1)$ $\quad N_2=\frac{1}{2}\xi(\xi+1)$ $\quad N_3=1-\xi^2$



$x=\sum_{i=1}^{3}x_iN_i$ (iso-parametric)

$$dx=\sum_{i=1}^{3}x_i\frac{\partial N_i}{\partial \xi}=\left[x_1\left(\frac{2\xi-1}{2}\right)+x_2\left(\frac{2\xi+1}{2}\right)-2\xi x_3\right]d\xi=\frac{\ell}{2}d\xi$$

In general, $x_1=x_1$, $x_2=x_1+\ell$, and $x_3=x_1+\frac{\ell}{2}$.

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{d\xi}{dx} = \frac{2}{\ell}\frac{\partial N_i}{\partial \xi}$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \end{bmatrix} = \frac{2}{\ell}\begin{bmatrix} \xi - \frac{1}{2} & \xi + \frac{1}{2} & -2\xi \end{bmatrix}$$

$$K = \int_0^{\ell}\int_A B^T C B \, dA\, dx = \frac{2EA}{\ell}\int_{-1}^{1} \underbrace{\begin{bmatrix} (\xi-\frac{1}{2})^2 & \xi^2 - \frac{1}{4} & -2\xi(\xi-\frac{1}{2}) \\ & (\xi+\frac{1}{2})^2 & -2\xi(\xi+\frac{1}{2}) \\ \text{symm.} & & 4\xi^2 \end{bmatrix}}_{F(\xi)} d\xi$$

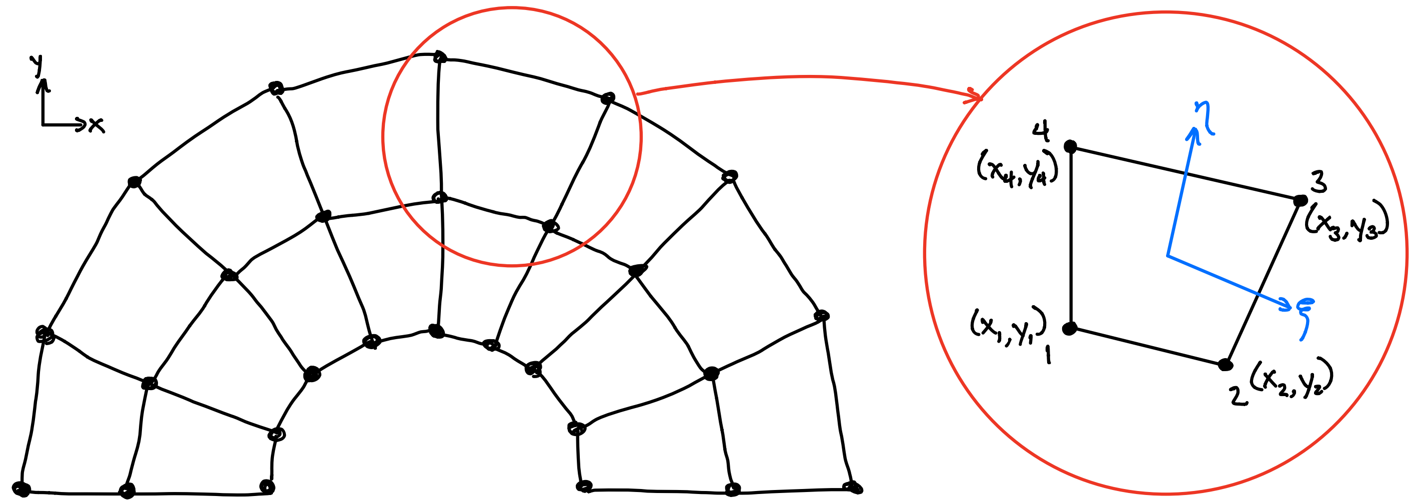
Since the highest polynomial order appearing in the matrix above is $q=2$, the rule states $2p-1 \geq q$ which means $p=2$.

Using the 2-point Gauss rule...

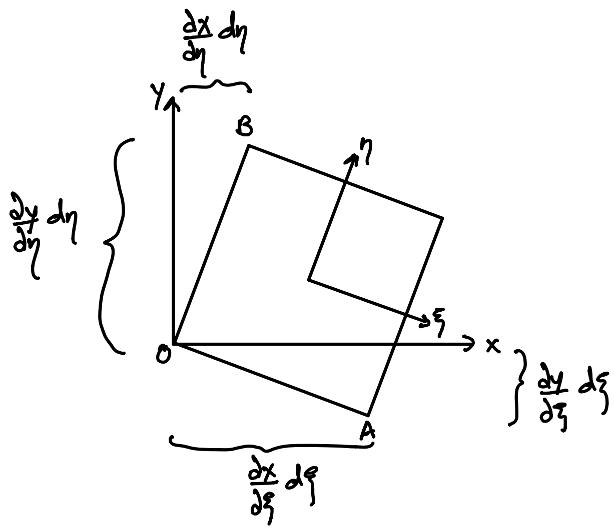
$$K = \frac{2EA}{\ell}\left[F(-1/\sqrt{3}) + F(1/\sqrt{3})\right] = \frac{EA}{3\ell}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}$$

In the previous, Gauss integration may not be necessary; however, it becomes very convenient as the elements become more elaborate - especially in higher dimensions where integration over multiple coordinates occurs.

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta)\,d\xi\,d\eta \approx \int_{-1}^{1}\sum_{i}^{p} \omega_i F(\xi_i,\eta)\,d\eta = \sum_{j}^{q}\sum_{i}^{p} \omega_i \tilde{\omega}_j F(\xi_i,\eta_j)$$



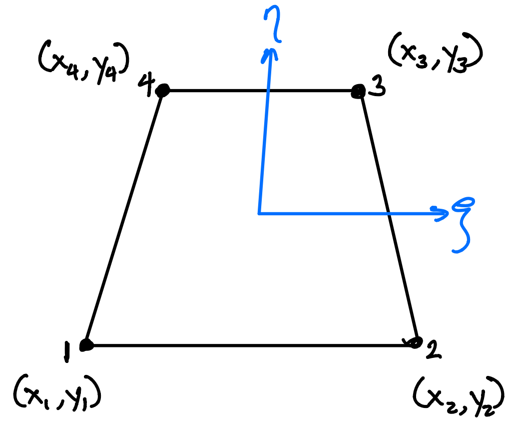
discretized system



$$d\vec{OA} = \frac{\partial x}{\partial \xi}\,d\xi\,\hat{i} + \frac{\partial y}{\partial \xi}\,d\xi\,\hat{j}$$

$$d\vec{OB} = \frac{\partial x}{\partial \eta}\,d\eta\,\hat{i} + \frac{\partial y}{\partial \eta}\,d\eta\,\hat{j}$$

$$dA = d\vec{OA} \times d\vec{OB} = \left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right)d\xi\,d\eta = |J|\,d\xi\,d\eta$$



$\xi \in [-1,1]\ ; \quad \eta \in [-1,1]$

$$\left.\begin{array}{ll} N_1 = \frac{1}{4}(1-\xi)(1-\eta) & N_2 = \frac{1}{4}(1+\xi)(1-\eta) \\ N_3 = \frac{1}{4}(1+\xi)(1+\eta) & N_4 = \frac{1}{4}(1-\xi)(1+\eta) \end{array}\right\}\text{4-node Quad}$$

STEP 0: Use the iso-parametric relation to map Cartesian to element coordinates.

$$x = \sum_{i=1}^{n} x_i N_i(\xi,\eta) \qquad y = \sum_{i=1}^{n} y_i N_i(\xi,\eta)$$

STEP 1: Take the derivative with respect to the element coordinates to determine the Jacobian.

$$\left.\begin{array}{ll} \frac{\partial x}{\partial \xi} = \sum_{i=1}^{n} x_i \frac{\partial N_i}{\partial \xi} & \frac{\partial y}{\partial \xi} = \sum_{i=1}^{n} y_i \frac{\partial N_i}{\partial y} \\ \frac{\partial x}{\partial \eta} = \sum_{i=1}^{n} x_i \frac{\partial N_i}{\partial \eta} & \frac{\partial y}{\partial \eta} = \sum_{i=1}^{n} y_i \frac{\partial N_i}{\partial \eta} \end{array}\right\} \rightarrow \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_2}{\partial \xi} & \cdots & \frac{\partial N_n}{\partial \xi} \\ \frac{\partial N_1}{\partial \eta} & \frac{\partial N_2}{\partial \eta} & \cdots & \frac{\partial N_n}{\partial \eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} = J$$

STEP 2: Use the chain rule to take the derivative of N_i with respect to the Cartesian coordinates.

$$\left.\begin{array}{l} \frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta}\frac{\partial \eta}{\partial y} \end{array}\right\} \rightarrow \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{J^{-1}} \begin{bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{bmatrix}$$

$\frac{\partial N_i}{\partial x}$ and $\frac{\partial N_i}{\partial y}$ now in terms of ξ and η

STEP 3: Use $N_{i,x}$ and $N_{i,y}$ to determine the strain-displacement matrix, B, and formulate the stiffness matrix integral.

$$B = \begin{bmatrix} N_{1,x} & 0 & N_{2,x} & 0 & \cdots & N_{n,x} & 0 \\ 0 & N_{1,y} & 0 & N_{2,y} & \cdots & 0 & N_{n,y} \\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & \cdots & N_{n,y} & N_{n,x} \end{bmatrix}$$

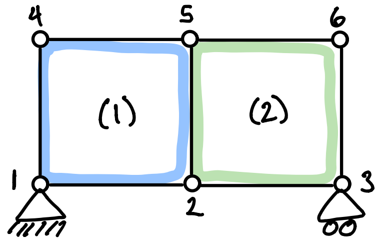
$$K^e = h\int_A B^T C B \, dA = h\int_{-1}^{1}\int_{-1}^{1} B^T C B |J| \, d\xi \, d\eta$$

STEP 4: Apply the Gauss integration rules to integrate the integral in K^e.

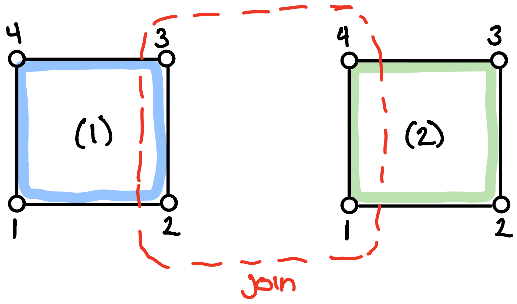
Table 17.1 - One-Dimensional Gauss Rules with 1 through 5 Sample Points

Points	Rule
1	$\int_{-1}^{1} F(\xi)\,d\xi \approx 2F(0)$
2	$\int_{-1}^{1} F(\xi)\,d\xi \approx F(-1/\sqrt{3}) + F(1/\sqrt{3})$
3	$\int_{-1}^{1} F(\xi)\,d\xi \approx \frac{5}{9}F(-\sqrt{3/5}) + \frac{8}{9}F(0) + \frac{5}{9}F(\sqrt{3/5})$
4	$\int_{-1}^{1} F(\xi)\,d\xi \approx w_{14}F(\xi_{14}) + w_{24}F(\xi_{24}) + w_{34}F(\xi_{34}) + w_{44}F(\xi_{44})$
5	$\int_{-1}^{1} F(\xi)\,d\xi \approx w_{15}F(\xi_{15}) + w_{25}F(\xi_{25}) + w_{35}F(\xi_{35}) + w_{45}F(\xi_{45}) + w_{55}F(\xi_{55})$

For the 4-point rule, $\xi_{34} = -\xi_{24} = \sqrt{(3-2\sqrt{6/5})/7}$, $\xi_{44} = -\xi_{14} = \sqrt{(3+2\sqrt{6/5})/7}$, $w_{14} = w_{44} = \frac{1}{2} - \frac{1}{6}\sqrt{5/6}$, and $w_{24} = w_{34} = \frac{1}{2} + \frac{1}{6}\sqrt{5/6}$.
For the 5-point rule, $\xi_{55} = -\xi_{15} = \frac{1}{3}\sqrt{5+2\sqrt{10/7}}$, $\xi_{45} = -\xi_{35} = \frac{1}{3}\sqrt{5-2\sqrt{10/7}}$, $\xi_{35} = 0$, $w_{15} = w_{55} = (322-13\sqrt{70})/900$, $w_{25} = w_{45} = (322+13\sqrt{70})/900$ and $w_{35} = 512/900$.



$u_{1x} = a = 0$
$u_{1y} = b = 0$
$u_{3y} = c = 0$



Local-Global (LG) Array

element

	(1)	(2)
1	1	2
2	2	3
3	5	6
4	4	5

local nodes   global nodes

Equation Array

global node

	1	2	3	4	5	6
u_x	a	1	3	4	6	8
u_y	b	2	c	5	7	9

DOF

From the Equation Array, we know that the final global stiffness matrix is 9x9, so we initialize a 9 x 9 zero matrix.

Each quad element has 8 DOF – two (u_x, u_y) for each node.

$$K^{(e)} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\ & & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ & & & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ & \text{symm.} & & & k_{55} & k_{56} & k_{57} & k_{58} \\ & & & & & k_{66} & k_{67} & k_{68} \\ & & & & & & k_{77} & k_{78} \\ & & & & & & & k_{88} \end{bmatrix}^{(e)}$$

Element (1):

local nodes	global nodes	equation (index)
1	1	a, b
2	2	1, 2
3	5	6, 7
4	4	4, 5

$$\begin{Bmatrix} a \\ b \\ 1 \\ 2 \\ 6 \\ 7 \\ 4 \\ 5 \end{Bmatrix} \{a \ b \ 1 \ 2 \ 6 \ 7 \ 4 \ 5\} =$$

$$\begin{bmatrix} aa & ab & a1 & a2 & a6 & a7 & a4 & a5 \\ & bb & b1 & b2 & b6 & b7 & b4 & b5 \\ & & 11 & 12 & 16 & 17 & 14 & 15 \\ & & & 22 & 26 & 27 & 24 & 25 \\ & \text{symm.} & & & 66 & 67 & 64 & 65 \\ & & & & & 77 & 74 & 75 \\ & & & & & & 44 & 45 \\ & & & & & & & 55 \end{bmatrix} \longleftrightarrow \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\ & & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ & & & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ & \text{symm.} & & & k_{55} & k_{56} & k_{57} & k_{58} \\ & & & & & k_{66} & k_{67} & k_{68} \\ & & & & & & k_{77} & k_{78} \\ & & & & & & & k_{88} \end{bmatrix}^{(1)}$$

cross out any rows with non-numeric index

only numeric indices contribute to global stiffness matrix

mixed indices contribute to the global force vector

Element (2):

local nodes	global nodes	equation (index)
1	2	1, 2
2	3	3, c
3	6	8, 9
4	5	6, 7

$$\begin{Bmatrix} 1 \\ 2 \\ 3 \\ c \\ 8 \\ 9 \\ 6 \\ 7 \end{Bmatrix} \{1 \ 2 \ 3 \ c \ 8 \ 9 \ 6 \ 7\} =$$

$$\begin{bmatrix} 11 & 12 & 13 & 1c & 18 & 19 & 16 & 17 \\ & 22 & 23 & 2c & 28 & 29 & 26 & 27 \\ & & 33 & 3c & 38 & 39 & 36 & 37 \\ & & & cc & c8 & c9 & c6 & c7 \\ & & & & 88 & 89 & 86 & 87 \\ & \text{symm.} & & & & 99 & 96 & 97 \\ & & & & & & 66 & 67 \\ & & & & & & & 77 \end{bmatrix} \longleftrightarrow \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} & k_{17} & k_{18} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} & k_{27} & k_{28} \\ & & k_{33} & k_{34} & k_{35} & k_{36} & k_{37} & k_{38} \\ & & & k_{44} & k_{45} & k_{46} & k_{47} & k_{48} \\ & & & & k_{55} & k_{56} & k_{57} & k_{58} \\ & \text{symm.} & & & & k_{66} & k_{67} & k_{68} \\ & & & & & & k_{77} & k_{78} \\ & & & & & & & k_{88} \end{bmatrix}^{(2)}$$

Global (assembled) stiffness matrix

$$
K = \begin{bmatrix}
k_{33}+k_{11} & k_{34}+k_{12} & k_{13} & k_{37} & k_{38} & k_{35}+k_{17} & k_{36}+k_{18} & k_{15} & k_{16} \\
 & k_{44}+k_{22} & k_{23} & k_{47} & k_{48} & k_{45}+k_{27} & k_{46}+k_{28} & k_{25} & k_{26} \\
 & & k_{33} & 0 & 0 & k_{37} & k_{38} & k_{35} & k_{36} \\
 & & & k_{77} & k_{78} & k_{57} & k_{67} & 0 & 0 \\
 & & & & k_{88} & k_{58} & k_{68} & 0 & 0 \\
 & & & & & k_{55}+k_{77} & k_{56}+k_{78} & k_{67} & k_{57} \\
 & & & & & & k_{66}+k_{88} & k_{58} & k_{68} \\
 & & & & & & & k_{55} & k_{56} \\
 & & & & & & & & k_{66}
\end{bmatrix}
$$

$$K^e = \int_{-1}^{1}\int_{-1}^{1} B^T C B |J| \, d\xi \, d\eta$$

let $C = \text{Identity}$ just for example.

$$= \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} (1-\xi^2)(1+\eta^2) & \cdots & & \cdot \\ \vdots & \ddots & & \\ & & \ddots & \\ \vdots & & & \ddots \end{bmatrix}_{8\times 8} d\xi \, d\eta$$

Each Quad element stiffness matrix is 8×8 and each term in the matrix changes depending on the (x,y) location of the element corners. I just made up the k_{11} term as an example.

Each term in K^e requires Gauss integration. For k_{11}, this is as follows:

$$k_{11} = \int_{-1}^{1}\int_{-1}^{1} (1-\xi^2)(1+\eta^2) \, d\xi \, d\eta = \int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta) \, d\xi \, d\eta$$

This requires 2-point rule

Integrate ξ first:

$$k_{11} = \int_{-1}^{1} [F(\xi = -1/\sqrt{3}, \eta) + F(\xi = 1/\sqrt{3}, \eta)] \, d\eta = \int_{-1}^{1} \frac{4}{3}(1+\eta^2) \, d\eta$$

Now integrate η:

$$k_{11} = \int_{-1}^{1} \frac{4}{3}(1+\eta^2) \, d\eta = \int_{-1}^{1} F(\eta) \, d\eta = F(\eta = -1/\sqrt{3}) + F(\eta = 1/\sqrt{3}) = \frac{32}{9}$$

This integration is done for every term in K^e and every quad element in the FE model.

Now, using the product rule discussed earlier...

$$\int_{-1}^{1}\int_{-1}^{1} F(\xi,\eta) \, d\xi \, d\eta \approx \sum_{j}^{q} \sum_{i}^{p} \omega_i \tilde{\omega}_j F(\xi_i, \eta_j) = \omega_1 \tilde{\omega}_1 F(\xi_1, \eta_1) + \omega_1 \tilde{\omega}_2 F(\xi_1, \eta_2) + \omega_2 \tilde{\omega}_1 F(\xi_2, \eta_1) + \omega_2 \tilde{\omega}_2 F(\xi_2, \eta_2) = \frac{32}{9}$$

$$\bar{\sigma} = C\varepsilon = C D N \bar{u} = C B \bar{u}$$

$$\bar{\sigma}_{avg} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_{avg} = \frac{1}{A} \int_{-1}^{1}\int_{-1}^{1} C B \bar{u} |J| \, d\xi \, d\eta \qquad A = \int_{-1}^{1}\int_{-1}^{1} |J| \, d\xi \, d\eta$$

TOPIC 7:
Shape Function Recipe

Shape Function Rules:

1. Interpolation condition: Unity at node i and zero at all other nodes.

2. Compatibility condition: Displacement on a side (2D) or face (3D) is determined uniquely by the nodes on that side/face.

3. Completeness condition: Can represent exactly any displacement field that is a linear polynomial in x and y. If 1 and 2 are true, then it is sufficient to check $\sum N_i = 1$.

Node Elimination Method

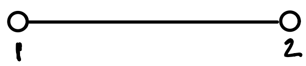
$N_i = c_i L_1 L_2 \cdots L_j$

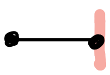
1. Find the minimum number of linear functions L_j that cross all nodes except i.

2. Set c_i such that $N_i = 1$ at i.

3. Check that the order of N_i on a side containing node i is 1 less than the number of nodes on that side. Verification ensures compatibility.

4. If completeness is satisfied, check that $\sum_i N_i = 1$.



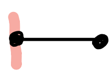


$N_1 = c_1 L_1$

$L_1 : \xi = 1 \rightarrow \xi - 1 = 0$

$N_1 = c_1(\xi - 1)$

$N_1(\xi = -1) = 1 \quad \therefore c_1 = -\frac{1}{2}$



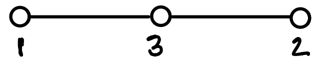
$N_2 = c_1 L_1$

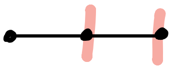
$L_1 : \xi = -1 \rightarrow \xi + 1 = 0$

$N_2 = c_1(\xi + 1)$

$N_2(\xi = 1) = 1 \quad \therefore c_1 = \frac{1}{2}$

$$\sum_{i=1}^{2} N_i = \frac{1}{2}(1-\xi) + \frac{1}{2}(1+\xi) = 1$$





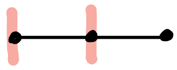
$N_1 = c_1 L_1 L_2$

$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

$L_2: \xi = 0$

$N_1 = c_1 \xi(\xi - 1)$

$N_1(\xi = -1) = 1 \quad \therefore \; c_1 = \frac{1}{2}$



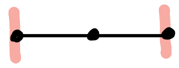
$N_2 = c_1 L_1 L_2$

$L_1: \xi = -1 \rightarrow \xi + 1 = 0$

$L_2: \xi = 0$

$N_2 = c_1 \xi(\xi + 1)$

$N_2(\xi = 1) = 1 \quad \therefore \; c_1 = \frac{1}{2}$



$N_3 = c_1 L_1 L_2$

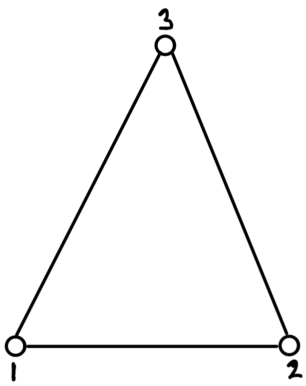
$L_1: \xi = -1 \rightarrow \xi + 1 = 0$

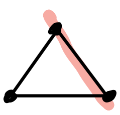
$L_2: \xi = 1 \rightarrow \xi - 1 = 0$

$N_3 = c_1(\xi^2 - 1)$

$N_3(\xi = 0) = 1 \quad \therefore \; c_1 = -1$

$$\sum_{i=1}^{3} N_i = \frac{1}{2}\xi(\xi - 1) + \frac{1}{2}\xi(\xi + 1) + 1 - \xi^2 = 1$$



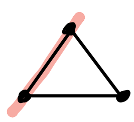


$N_1 = c_1 L_1$

$L_1: \xi_1 = 0$

$N_1 = c_1 \xi_1$

$N_1(\xi_1 = 1) = 1 \quad \therefore \; c_1 = 1$

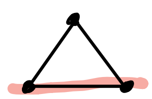


$N_2 = c_1 L_1$

$L_1: \xi_2 = 0$

$N_2 = c_1 \xi_2$

$N_2(\xi_2 = 1) = 1 \quad \therefore \; c_1 = 1$



$N_3 = c_1 L_1$

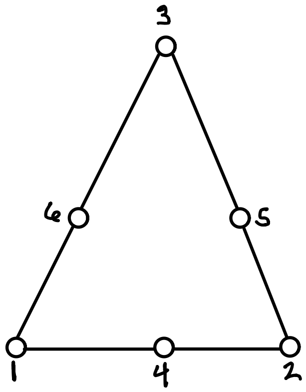
$L_1: \xi_3 = 0$

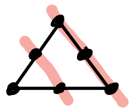
$N_3 = c_1 \xi_3$

$N_3(\xi_3 = 1) = 1 \quad \therefore \; c_1 = 1$

$$\sum_{i=1}^{3} N_i = \xi_1 + \xi_2 + \xi_3 = 1$$

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}$$





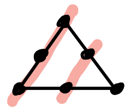
$N_1 = c_1 L_1 L_2$

$L_1: \xi_1 = 0$

$L_2: \xi_1 = \frac{1}{2} \rightarrow \xi_1 - \frac{1}{2} = 0$

$N_1 = c_1 \xi_1 \left(\xi_1 - \frac{1}{2}\right)$

$N_1(\xi_1 = 1) = 1 \quad \therefore \quad c_1 = 2$



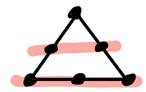
$N_2 = c_1 L_1 L_2$

$L_1: \xi_2 = 0$

$L_2: \xi_2 = \frac{1}{2} \rightarrow \xi_2 - \frac{1}{2} = 0$

$N_2 = c_1 \xi_2 \left(\xi_2 - \frac{1}{2}\right)$

$N_2(\xi_2 = 1) = 1 \quad \therefore \quad c_1 = 2$



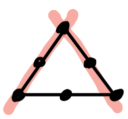
$N_3 = c_1 L_1 L_2$

$L_1: \xi_3 = 0$

$L_2: \xi_3 = \frac{1}{2} \rightarrow \xi_3 - \frac{1}{2} = 0$

$N_3 = c_1 \xi_3 \left(\xi_3 - \frac{1}{2}\right)$

$N_3(\xi_3 = 1) = 1 \quad \therefore \quad c_1 = 2$



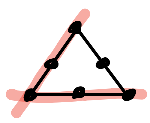
$N_4 = c_1 L_1 L_2$

$L_1: \xi_1 = 0$

$L_2: \xi_2 = 0$

$N_4 = c_1 \xi_1 \xi_2$

$N_4(\xi_1 = \frac{1}{2}, \xi_2 = \frac{1}{2}) = 1 \quad \therefore \quad c_1 = 4$



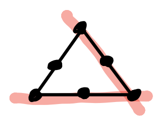
$N_5 = c_1 L_1 L_2$

$L_1: \xi_2 = 0$

$L_2: \xi_3 = 0$

$N_5 = c_1 \xi_2 \xi_3$

$N_5(\xi_2 = \frac{1}{2}, \xi_3 = \frac{1}{2}) = 1 \quad \therefore \quad c_1 = 4$



$N_6 = c_1 L_1 L_2$

$L_1: \xi_1 = 0$

$L_2: \xi_3 = 0$

$N_6 = c_1 \xi_1 \xi_3$

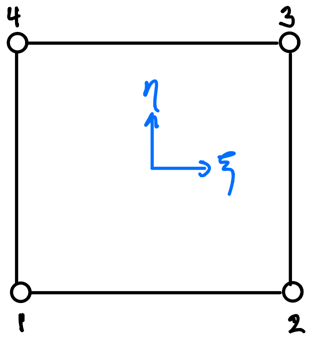
$N_6(\xi_1 = \frac{1}{2}, \xi_3 = 0) = 1 \quad \therefore \quad c_1 = 4$

$$\sum_{i=1}^{6} N_i = 2\xi_1\left(\xi_1 - \frac{1}{2}\right) + 2\xi_2\left(\xi_2 - \frac{1}{2}\right) + 2\xi_3\left(\xi_3 - \frac{1}{2}\right) + 4\xi_1\xi_2 + 4\xi_2\xi_3 + 4\xi_1\xi_3$$

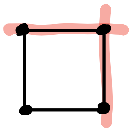
$$= 2\xi_1^2 + 2\xi_2^2 + 2\xi_3^2 - \cancel{(\xi_1 + \xi_2 + \xi_3)}^{\,1} + 4\xi_1\xi_2 + 4\xi_2\xi_3 + 4\xi_1\xi_3$$

$$= 2(\xi_1 + \xi_2 + \xi_3)^2 - 1 = 1$$

since $\xi_1 + \xi_2 + \xi_3 = 1$ by definition



$-1 \le \xi \le 1$
$-1 \le \eta \le 1$



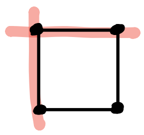
$N_1 = c_1 L_1 L_2$

$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

$L_2: \eta = 1 \rightarrow \eta - 1 = 0$

$N_1 = c_1(\xi - 1)(\eta - 1)$

$N_1(\xi = -1, \eta = -1) = 1 \quad \therefore \quad c_1 = \frac{1}{4}$



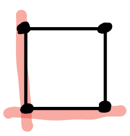
$N_2 = c_1 L_1 L_2$

$L_1: \xi = -1 \rightarrow \xi + 1 = 0$

$L_2: \eta = 1 \rightarrow \eta - 1 = 0$

$N_2 = c_1(\xi + 1)(\eta - 1)$

$N_2(\xi = 1, \eta = -1) = 1 \quad \therefore \quad c_1 = -\frac{1}{4}$



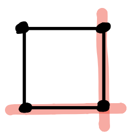
$N_3 = c_1 L_1 L_2$

$L_1: \xi = -1 \rightarrow \xi + 1 = 0$

$L_2: \eta = -1 \rightarrow \eta + 1 = 0$

$N_3 = c_1(\xi + 1)(\eta + 1)$

$N_3(\xi = 1, \eta = 1) = 1 \quad \therefore \quad c_1 = \frac{1}{4}$



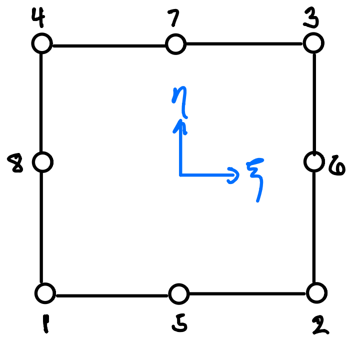
$N_4 = c_1 L_1 L_2$

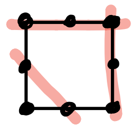
$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

$L_2: \eta = -1 \rightarrow \eta + 1 = 0$

$N_4 = c_1(\xi - 1)(\eta + 1)$

$N_4(\xi = -1, \eta = 1) = 1 \quad \therefore \quad c_1 = -\frac{1}{4}$





$N_1 = c_1 L_1 L_2 L_3$

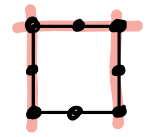
$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

$L_2: \eta = 1 \rightarrow \eta - 1 = 0$

$L_3: \xi + \eta + 1 = 0$

$N_1 = c_1(\xi - 1)(\eta - 1)(\xi + \eta + 1)$

$N_1(\xi = -1, \eta = -1) = 1 \quad \therefore \quad c_1 = -\frac{1}{4}$



$N_5 = c_1 L_1 L_2 L_3$

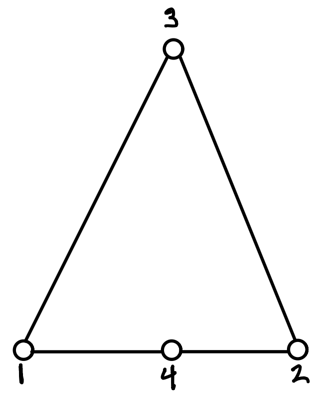
$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

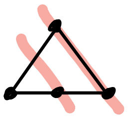
$L_2: \xi = -1 \rightarrow \xi + 1 = 0$

$L_3: \eta = 1 \rightarrow \eta - 1 = 0$

$N_5 = c_1(\xi^2 - 1)(\eta - 1)$

$N_5(\xi = 0, \eta = -1) = 1 \quad \therefore \quad c_1 = \frac{1}{2}$





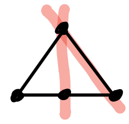
$N_1 = c_1 L_1 L_2$

$L_1: \xi_1 = 0$

$L_2: \xi_1 = \frac{1}{2} \rightarrow \xi_1 - \frac{1}{2} = 0$

$N_1 = c_1 \xi_1 (\xi_1 - \frac{1}{2})$

$N_1(\xi_1 = 1) = 1 \quad \therefore \quad c_1 = 2$



$N_1 = c_1 L_1 L_2$

$L_1: \xi_1 = 0$

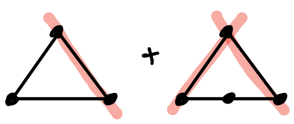
$L_2: \xi_1 = \xi_2 \rightarrow \xi_1 - \xi_2 = 0$

violates element coordinates definition since $\xi_1 + \xi_2 + \xi_3 \neq 1$

Along side 1-2, N_1 is order 2

Along side 1-3, N_1 is order 2 (too high; compatibility violated)

Systematic Approach: start by finding N_i for the parent element, and then add a correction.



$N_1 = \xi_1 + c_1 L_1 L_2$

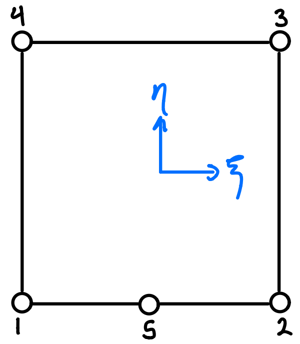
$L_1: \xi_1 = 0$

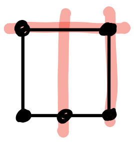
$L_2: \xi_2 = 0$

$N_1 = \xi_1 + c_1 \xi_1 \xi_2$

$N_1(\xi_1 = 1, \xi_2 = 0) = 1$

$N_1(\xi_1 = ½, \xi_2 = ½) = 0 \quad \therefore \quad c_1 = -2$





$N_1 = c_1 L_1 L_2 L_3$

$L_1: \xi = 0$

$L_2: \xi = 1 \rightarrow \xi - 1 = 0$

$L_3: \eta = 1 \rightarrow \eta - 1 = 0$

$N_1 = c_1 \xi(\xi-1)(\eta-1)$

$N_1(\xi=-1, \eta=-1) = 1 \quad \therefore \quad c_1 = -\frac{1}{4}$

$N_1 = c_1 L_1 L_2 L_3$

$L_1: \xi = 0$

$L_2: \xi = 1 \rightarrow \xi - 1 = 0$

$L_3: \eta = -2\xi - 1 \rightarrow \eta + 2\xi + 1 = 0$

$N_1 = c_1 \xi(\xi-1)(\eta+2\xi+1)$

$N_1(\xi=-1, \eta=-1) = 1 \quad \therefore \quad c_1 = -\frac{1}{4}$

Along side 1-2: $N_1(\xi, \eta=-1) = \frac{1}{2}\xi(\xi-1)$

Along side 1-4: $N_1(\xi=-1, \eta) = -\frac{1}{2}(\eta-1)$

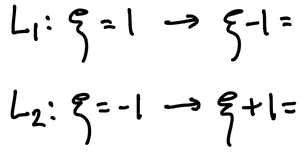
compatibility verified
lucky guess

Along side 1-2: $N_1(\xi, \eta=-1) = -\frac{1}{4}\xi(\xi-1)(2\xi+1)$

Along side 1-4: $N_1(\xi=-1, \eta) = -\frac{1}{2}(\eta-1)$

compatibility violated along side 1-2;
polynomial degree to high; bad guess

Systematic Approach:



$N_1 = \frac{1}{4}(\xi-1)(\eta-1) + c_1 L_1 L_2$

$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

$L_2: \xi = -1 \rightarrow \xi + 1 = 0$

$N_1 = \frac{1}{4}(\xi-1)(\eta-1) + c_1(\xi^2-1)$

$N_1(\xi=-1, \eta=-1) = 1$

$N_1(\xi=0, \eta=-1) = 0 \quad \therefore \quad c_1 = \frac{1}{2}$

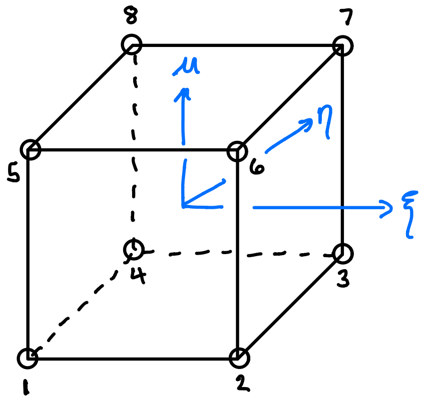
$N_5 = c_1 L_1 L_2$

$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

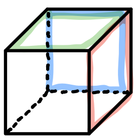
$L_2: \xi = -1 \rightarrow \xi + 1 = 0$

$N_5 = c_1(\xi^2 - 1)$

$N_5(\xi=0, \eta=-1) = 1 \quad \therefore \quad c_1 = -1$



$-1 \leq \xi \leq 1$
$-1 \leq \eta \leq 1$
$-1 \leq \mu \leq 1$



$N_1 = c_1 L_1 L_2 L_3$

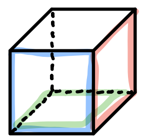
$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

$L_2: \eta = 1 \rightarrow \eta - 1 = 0$

$L_3: \mu = 1 \rightarrow \mu - 1 = 0$

$N_1 = c_1(\xi-1)(\eta-1)(\mu-1)$

$N_1(\xi=-1, \eta=-1, \mu=-1) = 1 \quad \therefore \quad c_1 = -\frac{1}{8}$



$N_8 = c_1 L_1 L_2 L_3$

$L_1: \xi = 1 \rightarrow \xi - 1 = 0$

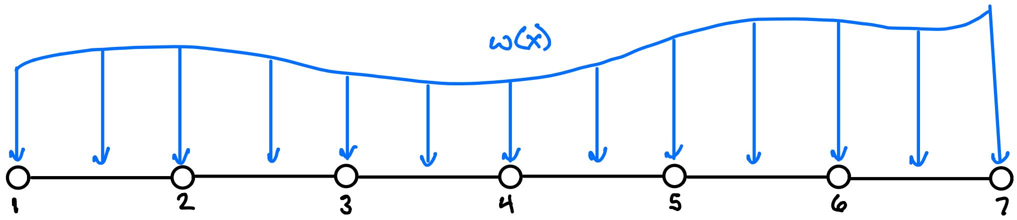
$L_2: \eta = -1 \rightarrow \eta + 1 = 0$

$L_3: \mu = -1 \rightarrow \mu + 1 = 0$

$N_8 = c_1(\xi-1)(\eta+1)(\mu+1)$

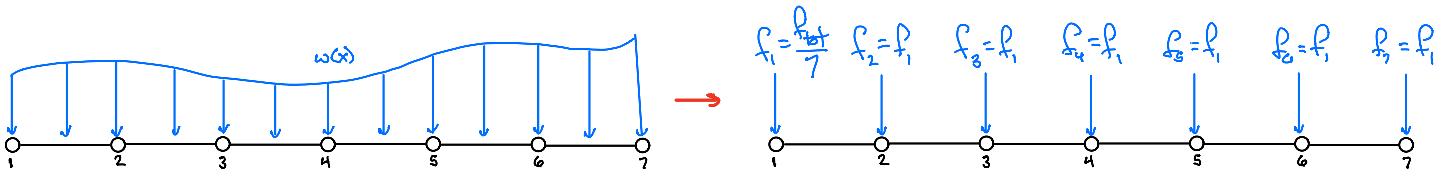
$N_8(\xi=-1, \eta=1, \mu=1) = 1 \quad \therefore \quad c_1 = \frac{1}{8}$

TOPIC 8:
Handling Distributed Loads



Direct Method: If the force variation over the structure is (1) gradual and (2) has a small amplitude (i.e., $\left|\frac{w_{max} - w_{min}}{w_{max}}\right| \le tol$ where the tolerance, tol, is small), then the total force f_{tot} is divided equally among all nodes.

$$f_{tot} = \int_0^{\ell} w(x)dx \quad \therefore \quad f_i = \frac{f_{tot}}{N} \quad \text{where } i = 1 \ldots N \quad (N: \text{\# of nodes})$$

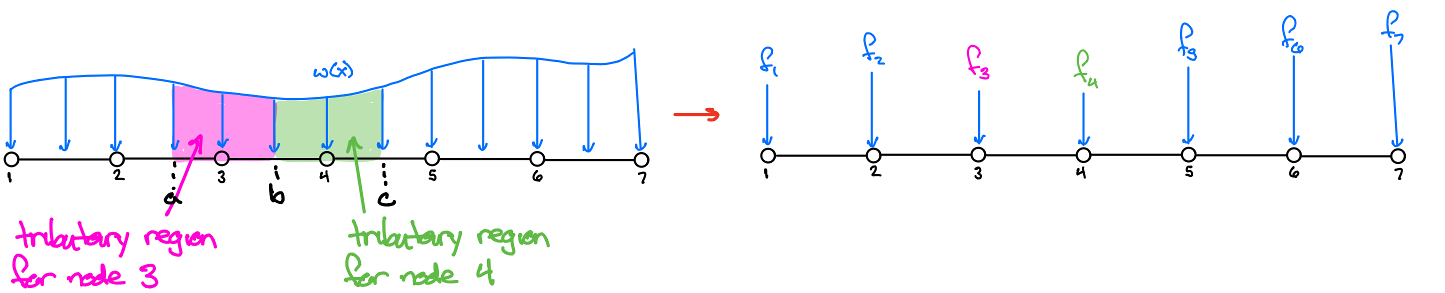


PROS: simple to implement.

CONS: not very accurate as it "smooths" out the force variations. Use this method only if you want a rough idea of the impact of a distributed load... or are lazy ☺.

Node-by-node Method: the force on node i is assigned from a tributary region about node i. The tributary region extends half-way to the neighboring nodes.

$$f = \int_a^b w(x)dx \quad (\text{total force over tributary region } x \in [a,b])$$

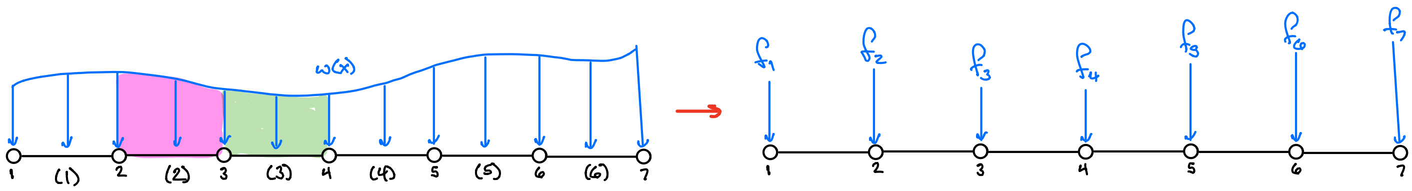


$$f_3 = \int_a^b w(x)dx \qquad f_4 = \int_b^c w(x)dx$$

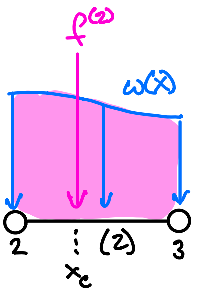
PROS: more accurate than the direct method.

CONS: requires knowledge of the force function w(x) and more calculation (i.e., integration).

Element-by-element Method: the force on node i is apportioned by statics from elements associated with node i

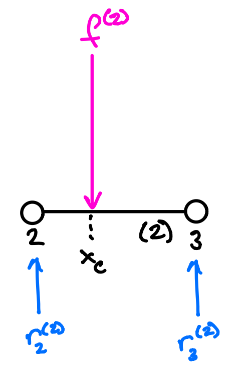


Node 3 is associated with elements (2) and (3). For element (2):



STEP 1: determine the total force over element (2) and the centroid where it acts.

$$f^{(2)} = \int_{x_2}^{x_3} \omega(x)\,dx \qquad x_c = \frac{\int_{x_2}^{x_3} \omega(x)\,x\,dx}{\int_{x_2}^{x_3} \omega(x)\,dx}$$



"reaction" forces; the negative of the applied forces

STEP 2: apportion the element force, $f^{(2)}$, to the nodes according to statics.

$$\sum F_y = -f^{(2)} + r_2^{(2)} + r_3^{(2)} = 0$$

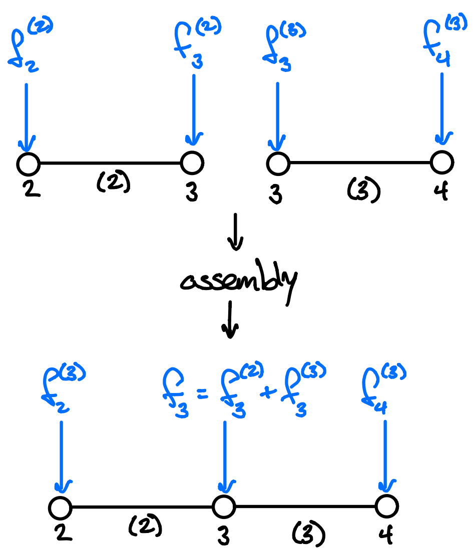
$$\sum M_2 = -f^{(2)}(x_c - x_2) + r_3^{(2)}\,\ell^{(2)} = 0$$

length of element (2), i.e., $\ell^{(2)} = x_3 - x_2$

$\therefore$ from $\sum M_2 = 0$, $r_3^{(2)} = \dfrac{f^{(2)}(x_c - x_2)}{\ell^{(2)}}$

$\therefore$ from $\sum F_y = 0$, $r_2^{(2)} = f^{(2)} - r_3^{(2)}$

$\therefore$ $f_2^{(2)} = -r_2^{(2)}$ and $f_3^{(2)} = -r_3^{(2)}$



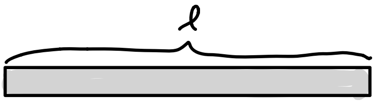
STEP 3: repeat STEPs 1-2 for every element connected to node 3. The total force at node 3 is the sum of contributions from each element.

PROS: more accurate than the node-by-node method since statics permits a more "fair" distribution of forces among nodes.

CONS: requires knowledge of the force function w(x) and more calculation (i.e., integration).

TOPIC 9:
Vibration of MDOF Systems

Direct Mass Matrix: the total mass of the element is apportioned equally to the nodes, ingoring any coupling. The result is a diagonal mass matrix, M.

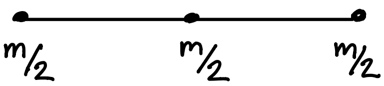


total mass: $m = \rho A l$

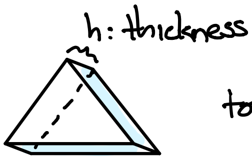
2-node rod

mass matrix: $M = \frac{1}{2}\rho A l \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acceleration: $\ddot{u} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}$

3-node rod

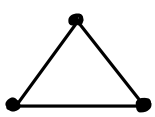


mass matrix: $M = \frac{1}{3}\rho A l \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ acceleration: $\ddot{u} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{bmatrix}$



total mass: $m = \rho A h$

3-node triangle



mass matrix: $M = \frac{1}{3}\rho A h \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ acceleration: $\ddot{u} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix}$

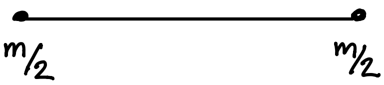
PROS: a diagonal mass matrix is more computationally efficient in terms of performing inverses as well as in storage (computer memory) if the sparsity is taken advantage of.

CONS: may account for either translational or rotational inertia, but not both (e.g., beams). May result in low-frequency dispersion in simulations.

Consistant (Variational) Mass Matrix: the kinetic energy is written in terms of the velocity vector, $v(x) = N\bar{v}$.

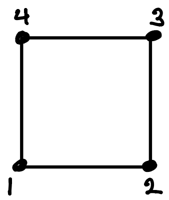
$$T = \frac{1}{2}\rho\int_V v^2\, dV = \frac{1}{2}\rho\int \bar{v}^T N^T N \bar{v}\, dV = \frac{1}{2}\bar{v}^T \underbrace{\rho\int_V N^T N\, dV}_{M}\, \bar{v} = \frac{1}{2}\bar{v}^T M \bar{v}$$

2-node rod



$$M = \rho\int_V N^T N\, dA\, dx = \frac{\rho A \ell}{2}\int_{-1}^{1} N^T N\, d\xi = \frac{\rho A \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Quad4



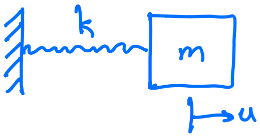
$$M = \rho\int_V N^T N\, dA\, dz = \rho h\int_{-1}^{1}\int_{-1}^{1} N^T N\, |J|\, d\xi\, d\eta$$

Jacobian depends on xy-location of nodes

PROS: both translational and rotational inertia are naturally accounted for. More representative of the physical system.

CONS: may result in low-frequency dispersion in simulations.

Before we tackle wave propagation, we will first consider selected topics on vibration, a phenomenon related to waves.



$\Sigma F = m\ddot{u} = -ku$ — the force of the spring opposes the displacement of the mass

$\ddot{u} + \omega_0^2 u = 0$ — ω_0: natural frequency

$\ddot{u} + \omega_0^2 u = 0 \longleftarrow u = Ae^{\alpha t}$

$(\alpha^2 + \omega_0^2)u = 0 \qquad \therefore \alpha = \pm i\omega_0$ — $\alpha^2 + \omega_0^2 = 0$, since $u(t) = 0$ is the trivial solution

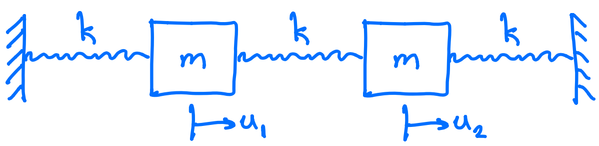
$u = A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}$ — for a linear problem, given two solutions, their sum is also a solution

$e^{\pm i\phi} = \cos\phi \pm i\sin\phi$ — Euler's formula

$u = (A_1 + A_2)\cos(\omega_0 t) + (A_1 - A_2)\sin(\omega_0 t)$

$= C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$ — C_1, C_2 determined from initial conditions

We've just solved an eigenvalue problem for the eigenvalue ω_0 (natural frequency) and eigenvector (mode shape) A. This may be better appreciated in MDOF systems.



$\Sigma F_1 = m\ddot{u}_1 = -2ku_1 + ku_2$

$\Sigma F_2 = m\ddot{u}_2 = ku_1 - 2ku_2$

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{M}\underbrace{\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix}}_{\ddot{\bar{u}}} + \underbrace{\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}_{K}\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\bar{u}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0} \longleftarrow \bar{u} = \bar{X}e^{i\omega t}$$

$$-\omega^2\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \longrightarrow \omega^2\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}^{-1}\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

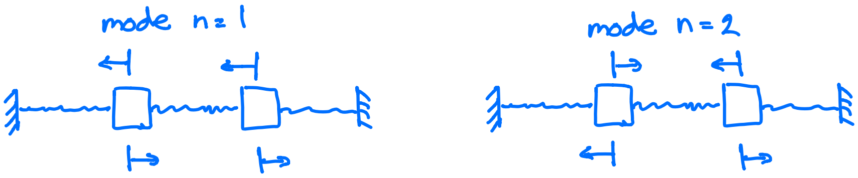
- matrix transforms (rotates or scales) a vector
- EVP asks what mode shape leads to a scaling transformation by factor ω^2?

$$\underbrace{\begin{bmatrix} 2k/m - \omega^2 & -k/m \\ -k/m & 2k/m - \omega^2 \end{bmatrix}}_{D:\ \text{dynamical matrix}}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

$|D| = m^2\omega^4 - 4km\omega^2 + 3k^2 = 0$ — characteristic equation

$\therefore \omega_1^2 = \frac{k}{m}$; $\omega_2^2 = \frac{3k}{m}$ — for an n-DOF system, there are at most n natural frequencies and mode shapes... degenerate frequencies are possible

$$\left(\frac{2k}{m}-\omega_n^2\right)X_1 - \frac{k}{m}X_2 = 0 \quad \therefore \quad \frac{X_2^{(n)}}{X_1^{(n)}} = \frac{2k - m\omega_n^2}{k} \longrightarrow \frac{X_2^{(1)}}{X_1^{(1)}} = 1 \; ; \; \frac{X_2^{(2)}}{X_1^{(2)}} = -1$$



- First mode: masses oscillate in phase.
- Second mode: masses oscillate 180° out of phase.
- Modes oscillate without forcing; $\omega \neq \omega_n$ or $X_i \neq X_i^{(n)}$ leaves remnant force.

$$X_1^{(1)} = X_2^{(1)} = 1 \; ; \; \omega = \omega_2$$

$$\left(\frac{2k}{m}-\omega_2^2\right)X_1^{(1)} - \frac{k}{m}X_2^{(1)} = \frac{2k-3k-k}{m} \neq 0 \qquad \text{nonzero force}$$

Now, we will consider an important property of normal modes — orthogonality. The natural frequency ω_n and modal vector $\bar{X}^{(n)}$ come as a pair and satisfy the equation:

$$\omega_i^2 M\bar{X}^{(i)} = K\bar{X}^{(i)} \qquad \text{mode } i$$

$$\omega_j^2 M\bar{X}^{(j)} = K\bar{X}^{(j)} \qquad \text{mode } j$$

Premultiply the previous by the transpose of the opposite mode:

$$\omega_i^2 [\bar{X}^{(j)}]^T M\bar{X}^{(i)} = [\bar{X}^{(j)}]^T K X^{(i)} \equiv [\bar{X}^{(i)}]^T K X^{(j)}$$

$$-\ \omega_j^2 [\bar{X}^{(i)}]^T M\bar{X}^{(j)} \equiv \omega_j^2 [\bar{X}^{(j)}]^T M \bar{X}^{(i)} = [\bar{X}^{(i)}]^T K X^{(j)}$$

equivalence due to symmetry of M and K

$$(\omega_i^2 - \omega_j^2)[\bar{X}^{(j)}]^T M\bar{X}^{(i)} = 0$$

In general, $\omega_i^2 \neq \omega_j^2$, therefore $[\bar{X}^{(j)}]^T M\bar{X}^{(i)} = 0$ and, by extension $[\bar{X}^{(j)}]^T K\bar{X}^{(i)} = 0$. Thus $X^{(i)}$ and $X^{(j)}$ are orthogonal with respect to both M and K. More directly,

$$\bar{X}^{(i)}[\bar{X}^{(j)}]^T M\bar{X}^{(i)} = \underbrace{\beta}_{\text{scalar}} \underbrace{M\bar{X}^{(i)}}_{\text{vector}} = \bar{0}$$

$M \neq 0$ and $\bar{X}^{(i)} = 0$ is the trivial solution, therefore $\beta = \bar{X}^{(i)}[\bar{X}^{(j)}]^T = 0$.

Due to their orthogonality, the mode shapes (eigenvectors) form a basis in n-dimensional space, meaning that any other vector must be some linear combination of these vectors.

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

basis in 3D rectilinear space

If $\bar{u}$ is an arbitrary vector in n-dimensional space, it can be expressed as:

$\bar{u} = \sum_{i=1}^{n} c_i \bar{X}^{(i)}$ c_i are constants; this is the expansion theorem

Premultiplying $\bar{u}$ by $[\bar{X}^{(i)}]^T M$ (or $[\bar{X}^{(i)}]^T K$) the value of c_i can be determined:

$$[\bar{X}^{(i)}]^T M \bar{u} = [\bar{X}^{(i)}]^T M \sum_{i=1}^{n} c_i \bar{X}^{(i)} = c_i [\bar{X}^{(i)}]^T M \bar{X}^{(i)} = c_i m_{ii} \qquad \therefore \; c_i = \frac{[\bar{X}^{(i)}]^T M \bar{u}}{m_{ii}}$$

drop the sum since $[\bar{X}^{(i)}]^T X^{(j)} = \delta_{ij}$ (orthogonality)

The expansion theorem is very useful in finding the response of MDOF systems subject to arbitrary forcing by decoupling the system of equations following a procedure called modal analysis.

Looking at $-\omega^2 M\bar{X} + K\bar{X} = 0$, it is apparent that the EOM are coupled by way of K. The coupling depends on the coordinates we use. It is possible to choose coordinates which uncouple the EOM.

$$\text{Let} \quad \bar{u} = \sum_{i=1}^{n} c_i \bar{X}^{(i)} = \underbrace{[\bar{X}^{(1)} \;\; \bar{X}^{(2)} \;\; \bar{X}^{(3)} \dots]}_{\text{transformation matrix}} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \end{bmatrix} = \mathbb{X}\bar{c}$$

coordinate vector (principal coordinates)

$$M\ddot{u} + Ku = M\mathbb{X}\ddot{\bar{c}} + K\mathbb{X}\bar{c} = 0$$

As we have seen, $[\bar{X}^{(i)}]^T M \bar{X}^{(i)} = m_{ii}$ and $[\bar{X}^{(i)}]^T K \bar{X}^{(i)} = k_{ii}$ diagonalize M and K. Let's premultiply the above by $\mathbb{X}^T$.

$$\mathbb{X}^T M \mathbb{X}\ddot{\bar{c}} + \mathbb{X}^T K \mathbb{X}\bar{c} = M_D\ddot{\bar{c}} + K_D\bar{c} = \overset{\text{modal mass}}{m_{ii}}\ddot{c}_i + \overset{\text{modal stiffness}}{k_{ii}} c_i = 0$$

since the equations are uncoupled, we can solve them one-by-one ... no complicated inverse or determinant needed.

$$m_{ii}\ddot{c}_i + k_{ii} c_i = 0 \longleftarrow \bar{c} = \bar{C}e^{i\omega t}$$

$$(-\omega^2 m_{ii} + k_{ii}) c_i = 0 \qquad \therefore \; \omega^2 = k_{ii}/m_{ii} = \omega_i^2$$

Recall our previous problem. Let's apply the expansion theorem/modal analysis.

$$\bar{X}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \quad \bar{X}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \therefore \; \mathbb{X} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

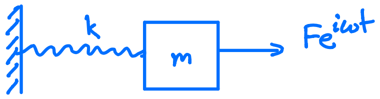
$$\mathbb{X}^T M \mathbb{X}\ddot{\bar{c}} + \mathbb{X}^T K \mathbb{X}\bar{c} = \begin{bmatrix} 2m & 0 \\ 0 & 2m \end{bmatrix}\begin{bmatrix} \ddot{c}_1 \\ \ddot{c}_2 \end{bmatrix} + \begin{bmatrix} 2k & 0 \\ 0 & 6k \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \longrightarrow \begin{cases} -2m\omega^2 c_1 + 2kc_1 = 0 \\ -2m\omega^2 c_2 + 6kc_2 = 0 \end{cases}$$

$$\therefore \; \omega^2 = \omega_1^2 = k/m$$

$$\therefore \; \underbrace{\omega^2 = \omega_2^2 = 3k/m}_{\text{just as before}}$$

The previous example is that of free vibration, i.e., the response of the system after an input of energy from initial conditions. Forcing represents a continuous input of energy.

Consider a SDOF system with a viscous damping element and harmonic forcing:



$m\ddot{u} + ku = Fe^{i\omega t}$ $\longleftarrow$ $u = Ae^{i\omega t}$ (steady-state response) ω: forcing frequency

$(-\omega^2 m + k)A = F$ $\longleftarrow$ divide by k and define $\delta_{st} = F/k$

$\therefore \left(-\frac{\omega^2}{\omega_0^2} + 1\right)A = \delta_{st}$ $\quad$ $\omega_0 = \sqrt{\frac{k}{m}}$: natural frequency (i.e., without forcing; from initial conditions only)

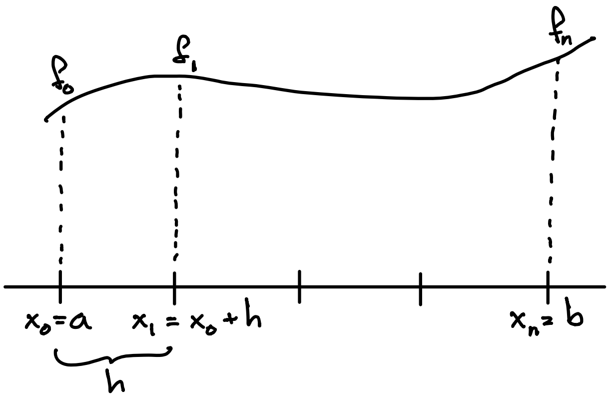
$\frac{A}{\delta_{st}} = \frac{1}{1-r^2}$ $\quad$ $r = \frac{\omega}{\omega_0}$

Degrees of freedom (DOFs): count the minimum number of independent coordinates necessary to describe the position of all parts of a system over time.

Discrete (the point-of-view of the FEM)	Continuous
finite # DOFs	infinite # DOFs
finite # natural freqs./shapes	infinite # natural freqs./mode shapes
formulated as Diff Eqs. (easy to solve)	formulated as PDEs (generally difficult to solve)
lumped/concentrated material parameters	distributed material parameters (i.e., functions of position)

TOPIC 10:
Finite Differences and Time Integration

The goal of the FDM is the replacement of the differential equation by a system of algebraic ones. First, we discretize the problem domain, and then substitute the derivatives with finite difference approximations.



x_i are called pivot points. f_i are the unknown values of the solution.

In general, the domain does not need to be divided into even intervals, but it is most convenient.

If $f(x)$ varies significantly between pivot point, the interval h should be reduced as part of convergence study.

A number of methods may be applied to approximate the derivatives; we chose the Taylor expansion about $x = x_0$:

$$f(x_0 \pm h) = f(x_0) \pm h f'(x_0) + \frac{h^2}{2} f''(x_0) \pm \frac{h^3}{3!} f'''(x_0) + \cdots + \frac{h^n}{n!} f^{(n)}(x_0) + \underbrace{O(h^{n+1})}_{\text{truncation error}}$$

Thus, for 1st order Taylor expansion with, we have:

$$f(x_0+h) = f(x_0) + h f'(x_0) + O(h^2) \quad \therefore \quad f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} = \frac{f_1 - f_0}{h} \qquad \text{(forward difference)}$$

$$f(x_0-h) = f(x_0) - h f'(x_0) + O(h^2) \quad \therefore \quad f'(x_0) = \frac{f(x_0) - f(x_0-h)}{h} = \frac{f_0 - f_{-1}}{h} \qquad \text{(backward difference)}$$

$$f(x_0+h) - f(x_0-h) = 2h f'(x_0) + O(h^2) \quad \therefore \quad f'(x_0) = \frac{f(x_0+h) - f(x_0-h)}{2h} = \frac{f_1 - f_{-1}}{2h} \qquad \text{(central difference)}$$

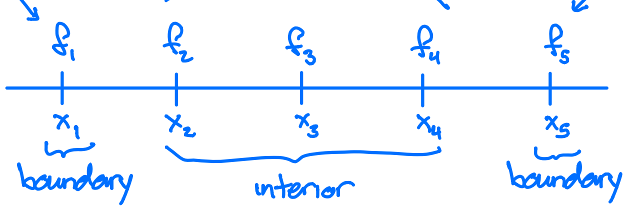
We can also formulate approximations at $O(h^4)$, but let's proceed with the central difference method to approximate $f''(x)$:

$$f(x_0+h) + f(x_0-h) = 2f(x_0) + h^2 f''(x_0) + O(h^2) \quad \therefore \quad f''(x_0) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

For an arbitrary pivot point x_i: $(f'')_i = \dfrac{f_{i+1} - 2f_i + f_{i-1}}{h^2}$

Using any of the difference schemes, we can approximate a derivative at pivot point x_i with $i = 1, \ldots, N$. In general, we can use these schemes as formulated; however, the boundaries $i = 1, N$ require special attention to accomodate the possibility of (x_0, f_0) and (x_{N+1}, f_{N+1}).

$$f'(x_1) = \frac{f_2 - f_0}{2h} \qquad f'(x_i) = \frac{f_{i+1} - f_{i-1}}{2h} \qquad f'(x_5) = \frac{f_6 - f_4}{2h}$$



f_0 and f_6 do not exist (are beyond the domain). To accommodate them, we generally have three options: periodic, Dirichlet, and Neumann (note: these are numerical BCs, not physical BCs).

Periodic: $f_0 = f_5$ and $f_6 = f_1$

Dirichlet: $f_0 = 0$ and $f_6 = 0$ (assumes solution decays to zero from the interior)

Neumann: $f_0 = f_1$ and $f_6 = f_5$ (assumes function varies linearly beyond the boundary, therefore, second-order and higher derivatives vanish.)

Other custom BCs may be applied as well.

For single-variable derivatives beyond second order and mixed derivatives of any order, there are a variety of alternative definitions. Consider, applying the first-order scheme twice in order to formulate a second order scheme.

$$\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{1}{2h}\left[\left(\frac{df}{dx}\right)_{i+1} - \left(\frac{df}{dx}\right)_{i-1}\right] = \frac{1}{2h}\left[\frac{df_{(i+1)}}{dx} - \frac{df_{(i-1)}}{dx}\right] = \frac{1}{4h}\left(f_{i+2} - 2f_i + f_{i-2}\right)$$

This scheme is less accurate than the one developed by Taylor series as function values further from the evaluation site are used to approximate the derivative. This hierarchal formulation is generally less accurate than the Taylor-derived version and becomes more so with increasing hierarchal levels. However, this is one option for mixed derivatives:

$$f(x_0+h, y_0+\ell) - f(x_0-h, y_0+\ell) = 2h f_x(x_0, y_0+\ell)$$

$$f(x_0+h, y_0-\ell) - f(x_0-h, y_0-\ell) = 2h f_x(x_0, y_0-\ell)$$

$$f_x(x_0, y_0+\ell) - f_x(x_0, y_0-\ell) = 2\ell f_{xy}(x_0, y_0)$$

$$\therefore\ f_{xy}(x_0, y_0) = \frac{f_x(x_0, y_0+\ell) - f_x(x_0, y_0-\ell)}{2\ell} = \frac{f(x_0+h, y_0+\ell) - f(x_0-h, y_0+\ell) - f(x_0+h, y_0-\ell) + f(x_0-h, y_0-\ell)}{4h\ell}$$

In other words $\displaystyle \frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{1}{2\ell}\left[\left(\frac{df}{dx}\right)_{j+1} - \left(\frac{df}{dx}\right)_{j-1}\right] = \frac{1}{2\ell}\left[\frac{df_{(j+1)}}{dx} - \frac{df_{(j-1)}}{dx}\right] = \frac{1}{4h\ell}\left(f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}\right)$

Notice that FD approximations are, essentially, weighted sums of function values in the vicinity of the pivot point. We can use this condition in addition to the Taylor series expansion to describe a general procedure for developing a FD approximation.

STEP 1: Determine the function values you wish to use in the approximation.

$f_i \quad f_{i-1} \quad f_{i+1} \quad f_{i+2}$ (stencil)

STEP 2: Determine the corresponding Taylor series expansion up to order n-1, where n = # function values.

$$
\left.\begin{aligned}
f_i &= f + 0\frac{\partial f}{\partial x} + 0\frac{\partial^2 f}{\partial x^2} + 0\frac{\partial^3 f}{\partial x^3}\\
f_{i-1} &= f - h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 f}{\partial x^3}\\
f_{i+1} &= f + h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 f}{\partial x^3}\\
f_{i+2} &= f + (2h)\frac{\partial f}{\partial x} + \frac{(2h)^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{(2h)^3}{6}\frac{\partial^3 f}{\partial x^3}
\end{aligned}\right\} \longrightarrow
\underbrace{\begin{bmatrix} f_i \\ f_{i-1} \\ f_{i+1} \\ f_{i+2} \end{bmatrix}}_{\substack{F \\ \text{function}}} =
\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & -h & \frac{h^2}{2} & -\frac{h^3}{6} \\ 1 & h & \frac{h^2}{2} & \frac{h^3}{6} \\ 1 & 2h & 2h^2 & \frac{4h^3}{3} \end{bmatrix}}_{\substack{W \\ \text{weights}}}
\underbrace{\begin{bmatrix} f \\ \frac{\partial f}{\partial x} \\ \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^3 f}{\partial x^3} \end{bmatrix}}_{\substack{D \\ \text{derivatives}}}
$$

$$
D = W^{-1}F \qquad \therefore \left(\frac{\partial f}{\partial x}\right)_i = \frac{-2f_{i-1} - 3f_i + 6f_{i+1} - f_{i+2}}{6h}
$$

$$
\left(\frac{\partial^3 f}{\partial x^3}\right)_i = \frac{-f_{i-1} + 3f_i - 3f_{i+1} + f_{i+2}}{h^3}
$$

Formulate a difference approximation of the mixed derivative $\frac{\partial^2 f}{\partial x \partial y}$ using $f_{i,j}$, $f_{i-1,j-1}$, $f_{i+1,j-1}$, $f_{i+1,j+1}$, and $f_{i-1,j+1}$.

$$
\left.\begin{aligned}
f_{i,j} &= f\\
f_{i-1,j-1} &= f - h\frac{\partial f}{\partial x} - \ell\frac{\partial f}{\partial y} + \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2h\ell\frac{\partial^2 f}{\partial x\partial y} + \ell^2\frac{\partial^2 f}{\partial y^2}\right)\\
f_{i+1,j-1} &= f + h\frac{\partial f}{\partial x} - \ell\frac{\partial f}{\partial y} + \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} - 2h\ell\frac{\partial^2 f}{\partial x\partial y} + \ell^2\frac{\partial^2 f}{\partial y^2}\right)\\
f_{i+1,j+1} &= f + h\frac{\partial f}{\partial x} + \ell\frac{\partial f}{\partial y} + \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2h\ell\frac{\partial^2 f}{\partial x\partial y} + \ell^2\frac{\partial^2 f}{\partial y^2}\right)\\
f_{i-1,j+1} &= f - h\frac{\partial f}{\partial x} + \ell\frac{\partial f}{\partial y} + \frac{1}{2}\left(h^2\frac{\partial^2 f}{\partial x^2} - 2h\ell\frac{\partial^2 f}{\partial x\partial y} + \ell^2\frac{\partial^2 f}{\partial y^2}\right)\\
f_{i+1,j} &= f + h\frac{\partial f}{\partial x} + \frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}
\end{aligned}\right\} \longrightarrow
\begin{bmatrix} f_{i,j} \\ f_{i-1,j-1} \\ f_{i+1,j-1} \\ f_{i+1,j+1} \\ f_{i-1,j+1} \\ f_{i+1,j} \end{bmatrix} =
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -h & -\ell & \frac{h^2}{2} & h\ell & \frac{\ell^2}{2} \\ 1 & h & -\ell & \frac{h^2}{2} & -h\ell & \frac{\ell^2}{2} \\ 1 & h & \ell & \frac{h^2}{2} & h\ell & \frac{\ell^2}{2} \\ 1 & -h & \ell & \frac{h^2}{2} & -h\ell & \frac{\ell^2}{2} \\ 1 & h & 0 & \frac{h^2}{2} & 0 & 0 \end{bmatrix}
\begin{bmatrix} f \\ \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial^2 f}{\partial x\partial y} \\ \frac{\partial^2 f}{\partial y^2} \end{bmatrix}
$$

$$
\therefore \frac{\partial^2 f}{\partial x\partial y} = \frac{f_{i-1,j-1} - f_{i+1,j-1} + f_{i+1,j+1} - f_{i-1,j+1}}{4h\ell}
$$

$$
\left.\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{f_{i-1,j-1} + f_{i-1,j+1} - 4f_i + 4f_{i+1,j} - f_{i+1,j-1} - f_{i+1,j+1}}{2h^2}\\
\frac{\partial^2 f}{\partial y^2} &= \frac{f_{i+1,j-1} - 2f_{i+1,j} + f_{i+1,j+1}}{\ell^2}
\end{aligned}\right\} \text{why?}
$$

Consider the governing equation of a homogeneous string (or rod/bar):

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad x \in (0, \ell) \qquad \text{governing equation}$$

$$\left.\begin{aligned} u(0,t) &= 0 \qquad t \in [0,T] \\ u(\ell,t) &= 0 \qquad t \in [0,T] \end{aligned}\right\} \text{boundary conditions}$$

$$\left.\begin{aligned} u(x,0) &= U(x) \qquad x \in (0,\ell) \\ \dot{u}(x,0) &= V(x) \qquad x \in [0,\ell] \end{aligned}\right\} \text{initial conditions}$$

1. Discretize the problem domain in space and time.

We discretize space into N_x points and time into N_t points, thus $\Delta x = \frac{\ell}{N_x}$ and $x_j = j\Delta x$ where $j = 0, \ldots, N_x$; similarly, $\Delta t = \frac{T}{N_t}$ and $t_k = k\Delta t$ with $k = 0, \ldots, N_t$.

2. Approximate derivatives by finite differences.

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{(\Delta x)^2} \qquad \frac{\partial^2 u}{\partial t^2} = \frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{(\Delta t)^2}$$

$$\frac{u_j^{k+1} - 2u_j^k + u_j^{k-1}}{(\Delta t)^2} = c^2\left[\frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{(\Delta x)^2}\right]$$

$$u_j^{k+1} - 2u_j^k + u_j^{k-1} = \left(\frac{c\Delta t}{\Delta x}\right)^2 (u_{j+1}^k - 2u_j^k + u_{j-1}^k)$$

$$u_j^{k+1} = \eta^2(u_{j+1}^k - 2u_j^k + u_{j-1}^k) - u_j^{k-1} + 2u_j^k \qquad (k = 1, \ldots, N_t)$$

$\eta = c\frac{\Delta t}{\Delta x}$ Courant (or CFL) number

$\Delta x/\Delta t$ is the "speed" limit of our numerical integration; therefore, it must be at least as fast as the physics of our problem (dictated by c), thus $\eta \le 1$ (stability condition).

3. Apply initial conditions ($k=0$).

$$\left.\frac{\partial u}{\partial t}\right|_{(x_j, t=0)} = \frac{u_j^1 - u_j^{-1}}{2\Delta t} = V(x_j) \qquad u = u_j^0 = U(x_j) \qquad (k=0)$$

$$u_j^{-1} = u_j^1 - 2\Delta t\, V(x_j)$$

$$u_j^1 = \frac{\eta^2}{2}(u_{j+1}^0 - 2u_j^0 + u_{j-1}^0) + \Delta t\, V(x_j) + u_j^0 \qquad (k=0)$$

4. Apply boundary conditions.

$$u(0,t) = u_j^k = 0 \quad (j=0) \qquad u(\ell,t) = u_j^k = 0 \quad (j = N_x)$$

```matlab
clear;clc;

%% Setup - User Input
n_site=1001; % # sites

L=1; % domain size
c=26; % speed of sound

T=3/100; % total simulation time
eta=1/3; % Courant #

%% Mesh Definition
x=linspace(0,L,n_site); % mesh
dx=x(2); % space step

dt=eta*dx/c; % time step
Nt=floor(T/dt); % # time steps

i_indx=[1:n_site 1:(n_site-1) 2:n_site];
j_indx=[1:n_site 2:n_site 1:(n_site-1)];
value=[-2*ones(1,n_site) ones(1,n_site-1) ones(1,n_site-1)];
K=sparse(i_indx,j_indx,value); % coefficient matrix

%% Explicit Time Integration
u_init=sparse(n_site,1); % initial displacement
v_init=sparse(n_site,1);v_init(round(n_site/2),1)=1; % initial velocity

u=sparse(n_site,Nt+1);u(:,1)=u_init; % displacement matrix; each column is
                                     % the displacement vector at different
                                     % time steps.
for k=1:Nt
    if k==1
        u(:,2)=(eta^2)*K*u(:,1)./2+u(:,1)+v_init*dt;
    else
        u(:,k+1)=(eta^2)*K*u(:,k)+2*u(:,k)-u(:,k-1);
    end

    %%%%% Plotting
    if floor(k/15)==k/15 % plot every 15 time steps
        plot(x,u(:,k+1),'k-','LineWidth',1);
        xlabel('Position, x');ylabel('Displacement, u');
        title(['t/T = ',num2str(round(k/Nt,3))]);
        axis([0 L -1e-5 3e-5]);
        drawnow;
    end
end
```

Newmark-β Integration Scheme
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1 Explicit Time Integration

Over the past couple lectures, we have developed the basics of numerical integration via the method of finite differences and even applied this tool toward the solution (i.e., simulation) of a string with a propagating disturbance. This example was a demonstration of an *explicit* integration scheme, i.e., one in which the solution at a later time step (i.e., u^{j+1}, $\dot{u}^{j+1}$, $\ddot{u}^{j+1}$) depends on the current conditions (i.e., u^j, $\dot{u}^j$, $\ddot{u}^j$) or earlier states. Now, let's consider a more general dynamic system than the string, one in which spatial discretization is accomplished via the finite element method, yielding:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{f}_{\text{int}}(\mathbf{u}) = \mathbf{f}_{\text{ext}}(t),$$

where $\mathbf{M}$ and $\mathbf{C}$ are, respectively, the mass and viscous damping matrices; $\mathbf{f}_{\text{int}}(\mathbf{u})$ and $\mathbf{f}_{\text{ext}}(t)$ are, respectively, the internal and time-depended (i.e., excitation) forces. Let's assume a linear internal force-displacement relation such that $\mathbf{f}_{\text{int}}(\mathbf{u}) = \mathbf{K}\mathbf{u}$ where $\mathbf{K}$ is the stiffness matrix. Thus,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}_{\text{ext}}(t). \tag{1}$$

Using central differences, we can replace the derivatives in Eq. (1) with the following approximations:

$$\dot{\mathbf{u}} = \dot{\mathbf{u}}^j = \frac{\mathbf{u}^{j+1} - \mathbf{u}^{j-1}}{2\Delta t}, \tag{2a}$$

$$\ddot{\mathbf{u}} = \ddot{\mathbf{u}}^j = \frac{\mathbf{u}^{j+1} - 2\mathbf{u}^j + \mathbf{u}^{j-1}}{(\Delta t)^2}, \tag{2b}$$

which, upon a few algebraic manipulations, gives the update rule

$$\left[\frac{1}{(\Delta t)^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C} + \mathbf{K}\right]\mathbf{u}^{j+1} = \frac{2}{(\Delta t)^2}\mathbf{M}\mathbf{u}^j - \left[\frac{1}{(\Delta t)^2}\mathbf{M} - \frac{1}{2\Delta t}\mathbf{C}\right]\mathbf{u}^{j-1} + \mathbf{f}_{\text{ext}}(j\Delta t), \tag{3}$$

with time step, Δt. Equation (3) represents a finite difference time-integration scheme. Notice that the displacement at a later time, $\mathbf{u}^{j+1}$, depends on current and earlier conditions, $\mathbf{u}^j$ and $\mathbf{u}^{j-1}$; therefore, the scheme is explicit.

Implementation Procedure:

1. Store the current conditions $\mathbf{u}^j$, $\dot{\mathbf{u}}^j$, and $\ddot{\mathbf{u}}^j$. If $j = 0$, then these are the initial conditions.
2. For a given integer $j \in [1, N_\text{t} - 1]$, solve the update rule for $\mathbf{u}^{j+1}$. N_t is the number of time steps.
3. Update the stored current conditions as $\mathbf{u}^{j+1} \to \mathbf{u}^j$, $\dot{\mathbf{u}}^{j+1} \to \dot{\mathbf{u}}^j$, and $\ddot{\mathbf{u}}^{j+1} \to \ddot{\mathbf{u}}^j$.
4. Repeat Steps 1–3 successively for all $j \in [1, N_\text{t} - 1]$.

There is, however, the small matter of the special case of $j = 0$ where, in Eq. (3), $j\Delta t = 0$ and $\mathbf{u}^{j-1} = \mathbf{u}^{-1}$. To resolve this issue, just as with the string problem, we use Eq. (2a) with $j = 0$ to determine that $\mathbf{u}^{-1} = \mathbf{u}^1 - 2\Delta t\dot{\mathbf{u}}^0$. Subsequent substitution into Eq. (1) with $j = 0$ gives

$$\left[\frac{2}{(\Delta t)^2}\mathbf{M} + \mathbf{K}\right]\mathbf{u}^1 = \frac{2}{(\Delta t)^2}\mathbf{M}\mathbf{u}^0 + \left[\frac{2}{\Delta t}\mathbf{M} - \mathbf{C}\right]\dot{\mathbf{u}}^0 + \mathbf{f}_{\text{ext}}(0).$$

After determining $\mathbf{u}^1$ for $j = 0$, the update rule in Eq. (3) is followed.

2 Implicit Time Integration

2.1 Linear Acceleration Scheme

In the derivation of the previous explicit, time integration scheme, we replaced the derivatives $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ with their finite difference approximations stemming from Taylor expansions. Now, let's try an alternative route. Assume the acceleration varies linearly over a given time increment, then:

$$\ddot{\mathbf{u}}(\tau) = \ddot{\mathbf{u}}^j + \frac{\tau - t_0}{\Delta t}(\ddot{\mathbf{u}}^{j+1} - \ddot{\mathbf{u}}^j) \qquad t_0 \le \tau \le t_0 + \Delta t,$$

where τ is a dummy variable. Next, we integrate the dummy variable, τ, over the time step, i.e., from t_0 to $t_0 + \Delta t$. Just as in undergraduate dynamics, we integrate once to arrive at the velocity and a second time to determine the displacement:

$$\dot{\mathbf{u}}(t) = \dot{\mathbf{u}}^j + \int_{t_0}^{t} \ddot{\mathbf{u}}(\tau)\mathrm{d}\tau = \dot{\mathbf{u}}^j + (t - t_0)\ddot{\mathbf{u}}^j + \frac{(t - t_0)^2}{2\Delta t}(\ddot{\mathbf{u}}^{j+1} - \ddot{\mathbf{u}}^j),$$

$$\mathbf{u}(t) = \mathbf{u}^j + \int_{t_0}^{t} \dot{\mathbf{u}}(\tau)\mathrm{d}\tau = \mathbf{u}^j + (t - t_0)\dot{\mathbf{u}}^j + \frac{(t - t_0)^2}{2}\ddot{\mathbf{u}}^j + \frac{(t - t_0)^3}{6\Delta t}(\ddot{\mathbf{u}}^{j+1} - \ddot{\mathbf{u}}^j).$$

As we are interested in the value of these quantities after one time step has elapsed, in the above, we substitute $t \to t_0 + \Delta t$, yielding

$$\dot{\mathbf{u}}^{j+1} = \dot{\mathbf{u}}^j + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^{j+1} + \ddot{\mathbf{u}}^j), \tag{4a}$$

$$\mathbf{u}^{j+1} = \mathbf{u}^j + \Delta t\dot{\mathbf{u}}^j + \frac{(\Delta t)^2}{6}(\ddot{\mathbf{u}}^{j+1} + 2\ddot{\mathbf{u}}^j). \tag{4b}$$

We could stop here and insert these definitions into the Eq. (1) to develop an implicit update rule, but lets continue.

2.2 Average Acceleration Scheme

Now, let's assume a constant, averaged acceleration over the time increment Δt given by

$$\ddot{\mathbf{u}}(\tau) = \frac{1}{2}(\ddot{\mathbf{u}}^{j+1} + \ddot{\mathbf{u}}^j) \qquad t_0 \le \tau \le t_0 + \Delta t$$

Now, we integrate the dummy variable, τ, over the time step. Integrating once yields the velocity; yet again gives the displacement:

$$\dot{\mathbf{u}}(t) = \dot{\mathbf{u}}^j + \int_{t_0}^{t} \ddot{\mathbf{u}}(\tau)\mathrm{d}\tau = \dot{\mathbf{u}}^j + \frac{t - t_0}{2}(\ddot{\mathbf{u}}^{j+1} + \ddot{\mathbf{u}}^j),$$

$$\mathbf{u}(t) = \mathbf{u}^j + \int_{t_0}^{t} \dot{\mathbf{u}}(\tau)\mathrm{d}\tau = \mathbf{u}^j + (t - t_0)\dot{\mathbf{u}}^j + \frac{(t - t_0)^2}{4}(\ddot{\mathbf{u}}^{j+1} + \ddot{\mathbf{u}}^j).$$

Again, substituting $t \to t_0 + \Delta t$ into the previous gives:

$$\dot{\mathbf{u}}^{j+1} = \dot{\mathbf{u}}^{j} + \frac{\Delta t}{2}(\ddot{\mathbf{u}}^{j+1} + \ddot{\mathbf{u}}^{j}) \tag{5a}$$

$$\mathbf{u}^{j+1} = \mathbf{u}^{j} + \Delta t\dot{\mathbf{u}}^{j} + \frac{(\Delta t)^2}{4}(\ddot{\mathbf{u}}^{j+1} + \ddot{\mathbf{u}}^{j}) \tag{5b}$$

We could stop here and insert these definitions into the Eq. (1) to develop an implicit update rule, but lets continue.

2.3 The Newmark-β Scheme

For the Newmark-β scheme, we essentially combine the results of the linear and averaged acceleration techniques. Consider the displacement definitions in Eqs. (4b) and (5b). Each of these may be represented by a single equation with the aid of a tuning variable, $\beta \in [\frac{1}{6}, \frac{1}{4}]$:

$$\mathbf{u}^{j+1} = \mathbf{u}^{j} + \dot{\mathbf{u}}^{j}\Delta t + \frac{(\Delta t)^2}{2}\left[2\beta\ddot{\mathbf{u}}^{j+1} + (1-2\beta)\ddot{\mathbf{u}}^{j}\right], \tag{6}$$

where $\beta = 1/6$ and $\beta = 1/4$, respectively, yield the linear and average definitions. Intermediate values result in a scheme that is a little bit of both, and β determines whether the scheme is more of the linear acceleration assumption or the average acceleration assumption. Values outside this range are meaningless and may result in either an unstable or less accurate scheme.

For the velocity, equations (4a) and (5a) are identical; however, notice that if the $\ddot{\mathbf{u}}^{j+1}$ term were absent, then we would have an explicit scheme (since the updated condition would depend on current or earlier conditions). Introducing the variable, $\gamma \in \left[0, \frac{1}{2}\right]$, we can combine the explicit and implicit definitions:

$$\dot{\mathbf{u}}^{j+1} = \dot{\mathbf{u}}^{j} + \Delta t\left[\gamma\ddot{\mathbf{u}}^{j+1} + (1-\gamma)\ddot{\mathbf{u}}^{j}\right], \tag{7}$$

where $\gamma = 0$ and $\gamma = \frac{1}{2}$, respectively, yield the explicit and implicit definitions.

- **Linear Acceleration (Implicit)**, $(\beta, \gamma) = \left(\frac{1}{6}, \frac{1}{2}\right)$
- **Average Acceleration (Implicit)**, $(\beta, \gamma) = \left(\frac{1}{4}, \frac{1}{2}\right)$
- **Explicit**, $(\beta, \gamma) = (0, 0)$

Ultimately, we are interested in the updated displacement, $\mathbf{u}^{j+1}$, when solving Eq. (1). We will accomplish this following a series of substitutions. First, let's solve Eq. (6) for $\ddot{\mathbf{u}}^{j+1}$:

$$\ddot{\mathbf{u}}^{j+1} = \frac{1}{\beta(\Delta t)^2}(\mathbf{u}^{j+1} - \mathbf{u}^{j} - \Delta t\dot{\mathbf{u}}^{j}) - \left(\frac{1-2\beta}{2\beta}\right)\ddot{\mathbf{u}}^{j}, \tag{8}$$

and then substitute this into Eq. (7), giving:

$$\dot{\mathbf{u}}^{j+1} = \dot{\mathbf{u}}^{j} + \Delta t\left[\frac{\gamma}{\beta(\Delta t)^2}(\mathbf{u}^{j+1} - \mathbf{u}^{j} - \Delta t\dot{\mathbf{u}}^{j}) + \left(\frac{2\beta-\gamma}{2\beta}\right)\ddot{\mathbf{u}}^{j}\right]. \tag{9}$$

Now, according to Eqs. (8) and (9), $\dot{\mathbf{u}}^{j+1}$ and $\ddot{\mathbf{u}}^{j+1}$ are each a function of $\mathbf{u}^{j+1}$. As we have done before, we replace the derivatives in Eq. (1) with these definitions and manipulate the result into the form below:

$$\left[\frac{1}{\beta(\Delta t)^2}\mathbf{M} + \frac{\gamma}{\beta\Delta t}\mathbf{C} + \mathbf{K}\right]\mathbf{u}^{j+1} = \left[\frac{1}{\beta(\Delta t)^2}\mathbf{M} + \frac{\gamma}{\beta\Delta t}\mathbf{C}\right]\mathbf{u}^j + \left[\frac{1}{\beta\Delta t}\mathbf{M} + \left(\frac{\gamma-\beta}{\beta}\right)\mathbf{C}\right]\dot{\mathbf{u}}^j + \dots$$
$$\left[\left(\frac{1-2\beta}{2\beta}\right)\mathbf{M} + \Delta t\left(\frac{\gamma-2\beta}{2\beta}\right)\mathbf{C}\right]\ddot{\mathbf{u}}^j + \mathbf{f}_{\mathrm{ext}}(j\Delta t). \tag{10}$$

Notice that, here, we need not concern ourselves with $j = 0$ as a special case!

Implementation Procedure:

1. For $j = 0$, initialize $\mathbf{u}^0$, $\dot{\mathbf{u}}^0$, and $\ddot{\mathbf{u}}^0$.
2. For a given integer $j \in [1, N_{\mathrm{t}} - 1]$, solve the update rule for $\mathbf{u}^{j+1}$. Then, substitute $\mathbf{u}^{j+1}$ into Eqs. (8) and (9) in order to determine $\ddot{\mathbf{u}}^{j+1}$ and $\dot{\mathbf{u}}^{j+1}$, respectively.
3. Update the stored current conditions as $\mathbf{u}^{j+1} \to \mathbf{u}^j$, $\dot{\mathbf{u}}^{j+1} \to \dot{\mathbf{u}}^j$, and $\ddot{\mathbf{u}}^{j+1} \to \ddot{\mathbf{u}}^j$.
4. Repeat Steps 2–3 successively for all $j \in [1, N_{\mathrm{t}} - 1]$.

Remark: In formulating our update rules, we have assumed the internal forces to be linearly proportional to the displacement, $\mathbf{f}_{\mathrm{int}}(\mathbf{u}) = \mathbf{K}\mathbf{u}$. If this is not the case (i.e., the force-displacement relation is non-linear), then a root-finding scheme (e.g., the Newton–Raphson method) will have to be employed to find the $\mathbf{u}^{j+1}$ which balances the equality in the update rule [Eq. (10)]. The form the rule takes will be essentially unchanged, e.g., the Newmark-β above would simply become:

$$\left[\frac{1}{\beta(\Delta t)^2}\mathbf{M} + \frac{\gamma}{\beta\Delta t}\mathbf{C}\right]\mathbf{u}^{j+1} + \mathbf{f}_{\mathrm{int}}\left(\mathbf{u}^{j+1}\right) = \text{Same Right-Hand Side.}$$